Waves and Instabilities in Inhomogeneous Plasmas

Cours 3ème Cycle

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0.1 Introduction

A plasma in thermodynamic equilibrium is characterised by homogeneous Maxwellian particle distributions at rest with each other. Deviations of the plasma from such a thermodynamic equilibrium state can be the source of free energy, leading, under certain conditions, to instabilities. Such deviations can arise both in an homogeneous or an inhomogeneous system.

In an homogeneous system, deviations from thermodynamic equilibrium are found in velocity space. The ion-acoustic instability is an example of an instability arising from such a velocity deviation, in this particular case in the form of electrons streaming with respect to ions. The Weibel instability is another example, which results from an anisotropic electron velocity distribution. These two illustrations are described in the lecture notes by K. Appert[1].

The present notes address the instabilities arising in inhomogeneous plasmas. Inhomogeneities can only be maintained for a certain length of time by trapping the plasma in some way. Magnetic fields are an obvious choice for achieving this purpose, and the following discussion is thus limited to the analysis of magnetised plasmas.

Despite the constraints imposed on the plasma by the magnetic fields, the confinement of inhomogeneities in any case deteriorates ultimately through ordinary transport processes involving collisions. Indeed, collisions of each species (electrons, ions) with itself, as well as collisions of different species with each other, lead to homogeneous Maxwellian distributions with zero relative average velocities. In time, a uniform state of thermodynamic equilibrium is reached in this way. However, the instabilities that may arise in such inhomogeneous systems often provide a more efficient channel through which the particle and energy confinement deteriorates.

Indeed, instabilities arising in inhomogeneous plasmas can be the origin of a turbulent state characterised by a certain level of fluctuations. The electromagnetic fields associated with these fluctuations can cause stochastic motion of the constituent plasma particles. This motion leads to so-called anomalous transport, and results in the escape of particles and energy from the system. The heat and particle loss observed in most plasma confinement experiments are mainly attributed to this mechanism of plasma turbulence.

These notes concentrate on the instabilities at the origin of this turbulent transport, i.e. the class of so-called microinstabilities, which are the set of low-frequency modes, subsisting even when the large-scale magnetohydrodynamic modes have been suppressed.

Critical to the mechanism of these microinstabilities is the dissimilar dynamics of the different particle species in an inhomogeneous magnetised plasmas. A single fluid description of the plasma, such as used in magnetohydrodynamics, is thus not suited in this context, and at least a two fluid representation is required. In fact, many important features of these microinstabilities, such as wave-particle resonances and finite Larmor radius effects require a kinetic-type description.

Furthermore, the study of the instabilities is limited here to their onset (underlying mechanisms, critical conditions), and the following discussion is thus reduced to a linear analysis. Also, the perturbations considered here are essentially electrostatic. This approximation is valid assuming a low β (= kinetic pressure / magnetic pressure) plasma.

Electron mass m_e	$9.11 \cdot 10^{-31}$ kg
Ion mass ($\sim$ proton) m_i	$1.67 \cdot 10^{-27}$ kg
Degree of ionisation Z	1
Magnetic field B	5 Tesla
Electron density N_e	$10^{21}\ m^{-3}$
Electron temperature T_e	10^4 eV
Ion temperature T_i	10^4 eV
Characteristic gradient length L	1 m

Table 1: Typical physical parameters of magnetic fusion-type plasmas

Electron plasma frequency ω_{pe}	$1.78 \cdot 10^{12}\ s^{-1}$
Electron cyclotron frequency Ω_{ce}	$8.78 \cdot 10^{11}\ s^{-1}$
Ion cyclotron frequency Ω_{ci}	$4.79 \cdot 10^{8}\ s^{-1}$
Electron thermal velocity $v_{th\,e}$	$4.19 \cdot 10^{7}$ m/s
Ion thermal velocity $v_{th\,i}$	$9.79 \cdot 10^{5}$ m/s
Electron Debye Length λ_{De}	$2.35 \cdot 10^{-5}$ m
Ion Debye Length λ_{Di}	$2.35 \cdot 10^{-5}$ m
Electron Larmor radii $\lambda_{L\,e}$	$4.77 \cdot 10^{-5}$ m
Ion Larmor radii $\lambda_{L\,i}$	$2.04 \cdot 10^{-3}$ m
Electron-ion collision freq. ν_{ei}	$\sim 1 \cdot 10^{5}\ s^{-1}$
Drift frequency $\omega^\star \sim v_{th\,i}/L$	$\sim 1 \cdot 10^{6}\ s^{-1}$

Table 2: Corresponding time and length scales

Finally, the limit of an ideal plasma will be assumed so that the effect of collisions on the instabilities is neglected.

The emphasis here is on instabilities in magnetic fusion-type plasmas. Throughout these notes one therefore assumes a plasma formed by a single ion species. Also, one assumes that these ions are singly ionised ($Z = 1$), it being understood that the main interest is for the case of a hydrogen (deuterium, tritium) plasma. Typical magnetic fusion-type parameters are given in Table 1. As a reference, the different corresponding characteristic time and length scales are given in Table 2. These tables also define notations for the various physical quantities. Note in particular the scalings $\lambda_{L\,e} \ll \lambda_{L\,i} \ll L$, $\lambda_{De,i} \ll \lambda_{L\,i}$, and $\omega^\star \ll \Omega_{ce,i}$, which will be extensively applied in the following derivations.

Chapter 1, dealing with instabilities in slab magnetic geometry is mainly inspired by references [2, 3, 4].

These notes can be downloaded in pdf format through the CRPP public web page:

`http://crppwww.epfl.ch/ brunner/inhomoplasma.pdf`.

The Matlab code for numerically solving the dispersion relations discussed in these notes is available on the CRPP SUN-cluster under directory:

`/home/da12/brunner/TEX/COURS_3EME_CYCLE/INHOMO_PLASMA/MODEL/LOCDISP`.

Chapter 1

Inhomogeneous Slab Systems

1.1 Drifts in Magnetic Fields

As will appear clearly further on, an important feature underlying the instabilities in inhomogeneous plasmas are the various drifts perpendicular to the magnetic field that can arise in a magnetised plasma. These drifts can be at the microscopic level, i.e. of the magnetised particles themselves submitted to some external force $\vec{F}$. These particle drifts can naturally lead to macroscopic drifts as well, defined as the average velocity over the whole particle distribution.

In inhomogeneous plasmas, such macroscopic drifts may in fact arise even for stationary gyro-centres (= centre of the Larmor rotation) of the particles. These are the so-called diamagnetic drifts.

1.1.1 Drift of Particles Submitted to an External Force

The equation of motion for a particle in a magnetic field $\vec{B}$, submitted to an additional force $\vec{F}$, assumed perpendicular to $\vec{B}$, is given by

$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B} + \vec{F},$$

where m and q are respectively the mass and charge of the particle, and $\vec{v}$ its velocity in the lab frame. Making the change of variables $\vec{w} = \vec{v} - \vec{v}_F$, where $\vec{v}_F$ is defined by

$$\vec{v}_F = \frac{\vec{F} \times \vec{B}}{qB^2}, \tag{1.1}$$

one obtains:

$$\begin{aligned} m\frac{d\vec{w}}{dt} &= m\frac{d}{dt}(\vec{v} - \vec{v}_F) = q(\vec{v}_F + \vec{w}) \times \vec{B} + \vec{F} \\ &= q\,\vec{w} \times \vec{B}, \end{aligned} \tag{1.2}$$

having used:

$$q\,\vec{v}_F \times \vec{B} = \frac{\vec{F} \times \vec{B}}{B^2} \times \vec{B} = -\vec{F}.$$

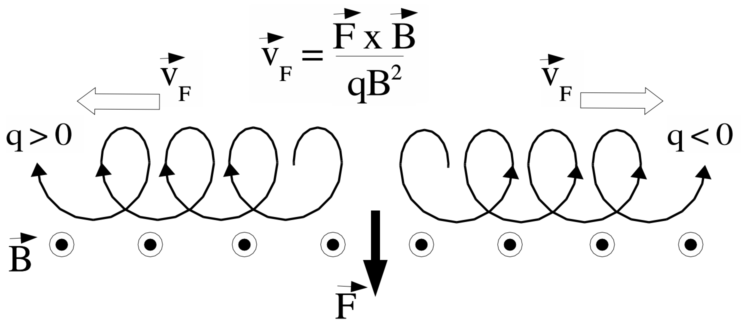

Figure 1.1: Drifts of particles in a magnetic field $\vec{B}$, submitted to an additional external force $\vec{F}$. If, as shown here, the force $\vec{F}$ is independent of the sign of the electric charge, electrons and ions drift in opposite directions. In all cases the drift is given by $\vec{v}_F = (\vec{F} \times \vec{B})/(qB^2)$.

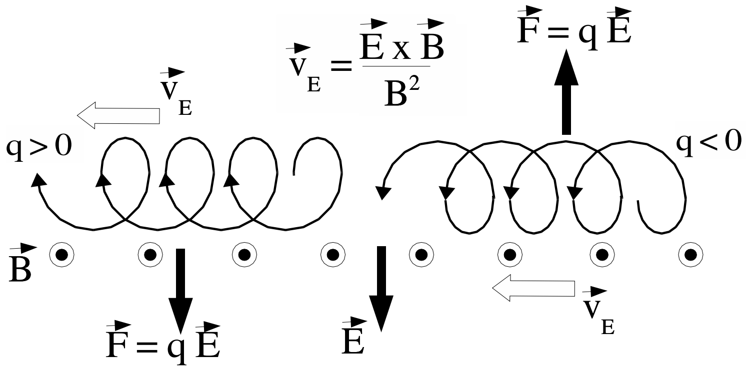

Figure 1.2: Particular case of a force depending on the sign of the electric charge: Particles submitted to an electric force $\vec{F} = q\vec{E}$. Here, all particles drift with the same velocity $\vec{v}_E = (\vec{E} \times \vec{B})/B^2$.

The equation (1.2) for $\vec{w}$ is the equation of motion of the particle in the magnetic field $\vec{B}$ alone. The motion of the particle in the lab frame is thus the superposition of a gyro-motion with cyclotron frequency $\Omega = qB/m$, and of a drift motion with velocity $\vec{v}_F$. Illustrations of such particle drifts are given in Figures 1.1 and 1.2.

1.1.2 Diamagnetic Drifts

A particular feature of a magnetised plasma is the presence of average drifts resulting from the interplay between spatial inhomogeneities and the finiteness of the Larmor radius. One must emphasise, that these so-called diamagnetic drifts do not arise from individual particle drifts as the ones discussed in the previous section.

These diamagnetic drifts can be described by considering a plasma in a uniform magnetic field $\vec{B}$, with no additional external force $\vec{F}$. In such a system there are indeed no

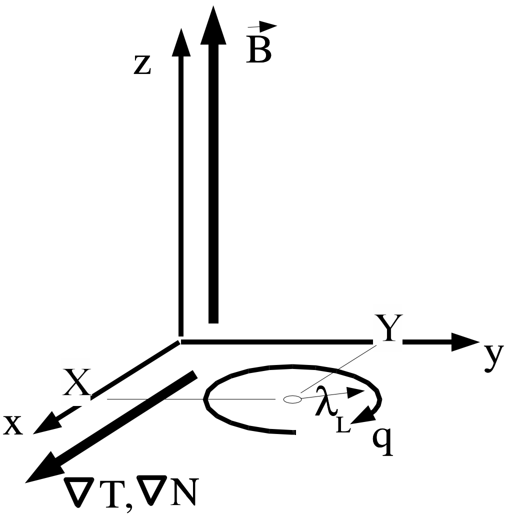

Figure 1.3: Slab geometry of a magnetised inhomogeneous plasma.

particle drifts. One starts by setting the orthonormal right handed system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, such that the magnetic field $\vec{B}$ is aligned along $\vec{e}_z$ (see Fig. 1.3). For the distribution function $f_0(\vec{r}, \vec{v})$ of a given species to represent a stationary state, i.e. to be a stationary solution to the Vlasov equation (assuming here no collisions), it must be function of the constants of motion. Hence, besides the dependence in the kinetic energy $\mathcal{E} = mv^2/2$, f_0 is chosen function of the position $X = x + v_y/\Omega$ along $\vec{e}_x$ of the Larmor rotation centre as well. This enables to define a quasi-Maxwellian distribution with both density and temperature inhomogeneities:

$$f_0(X, \mathcal{E}) = \frac{N(X)}{[2\pi T(X)/m]^{3/2}} \exp -\frac{\mathcal{E}}{T(X)}. \tag{1.3}$$

Indeed, to zero order in the Larmor radius $\lambda_L = v_\perp/\Omega$, this form is a Maxwellian distribution with density $N(x)$ and temperature $T(x)$. Assuming that the characteristic length $L \sim |d\ln N/dx|^{-1}, |d\ln T/dx|^{-1}$ of the inhomogeneities is large compared to the average Larmor radius λ_L of the particles ($\sim$ weak gradients), one can expand to first order in the Larmor radius:

$$f_0(X, \mathcal{E}) = f_0(x, \mathcal{E}) + \frac{v_y}{\Omega}\left(\frac{d\ln N}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right) f_0(x, \mathcal{E}) + \mathcal{O}(\epsilon^2),$$

having defined the small parameter $\epsilon = \lambda_L/L \ll 1$.

Integrating to obtain the average velocity gives:

$$\begin{aligned} \vec{V}_d &= \int d\vec{v}\,\vec{v} f_0(X, \mathcal{E}) / \int d\vec{v}\, f_0(X, \mathcal{E}) = \frac{1}{N}\left(\frac{d\ln N}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right)\int d\vec{v}\,\vec{v} f_0(x, \mathcal{E})\frac{v_y}{\Omega} \\ &= \frac{1}{qB}\frac{1}{N}\left(\frac{d\ln N}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right) NT\vec{e}_y = \frac{1}{qB}\frac{1}{N}\frac{d(NT)}{dx}\vec{e}_y = \frac{1}{qB^2}(-\frac{\nabla p}{N} \times \vec{B}). \end{aligned} \tag{1.4}$$

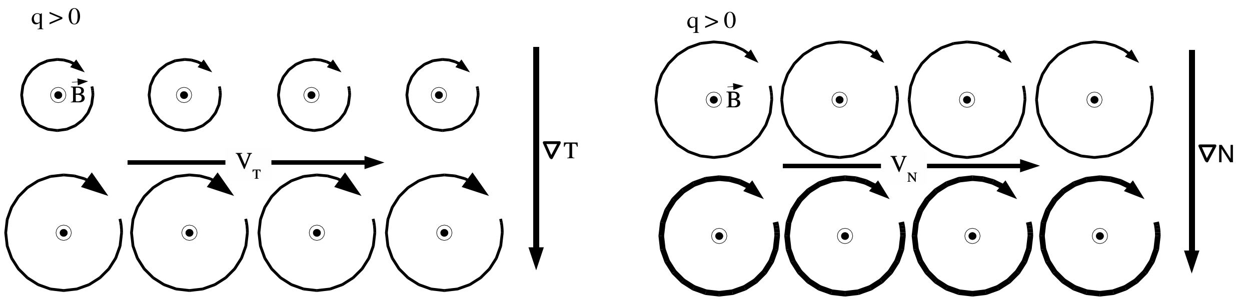

Figure 1.4: Diamagnetic drift resulting from temperature and density gradients

Note that (1.4) is again of the form (1.1) but here $\vec{F} = -\nabla p/N$ is the macroscopic force related to pressure gradients ∇p, with $p = NT$. Note, that for similar density and temperature gradients for electrons and ions, this force is essentially charge independent, so that the corresponding diamagnetic drifts are in opposite directions.

In the following section, a similar derivation will be carried out in somewhat more detail for a system containing both particle and diamagnetic drifts.

Exercise:

1. Derive diamagnetic drifts from a fluid-like representation.
2. Why are these drifts called "diamagnetic".

1.2 Dispersion Relation in Slab Geometry

As a basis for studying the instabilities that may arise in an inhomogeneous slab plasma, a general, local dispersion relation is now derived. One starts here by considering a plasma in a slab geometry, and applies the same methods used for computing the dispersion relation in an homogeneous magnetised plasma: Solving the linearized Vlasov equation by integrating along the unperturbed trajectories of the particles.

One considers the same slab system as represented in Fig.1.3, however assuming that the particles may also be submitted to an additional uniform, external force field $\vec{F} = F\vec{e}_x$ perpendicular to $\vec{B}$. This force field will naturally induce a drift motion along Oy on the magnetised particles of the system.

1.2.1 Equilibrium State

Assuming a collisionless plasma, the evolution of each species distribution $f(\vec{r}, \vec{v}, t)$ in the magnetic field $\vec{B}$ and external force field $\vec{F}$ is given by the following Vlasov equation:

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} + \frac{1}{m}\left(q\vec{v}\times\vec{B} + \vec{F}\right)\cdot\frac{\partial}{\partial \vec{v}}\right] f = 0.$$

To represent the equilibrium plasma, one must find a stationary solution $f = f_0 \neq f_0(t)$ to this Vlasov equation. A necessary and sufficient condition is that f_0 is a function of the constants of motion.

The constants of motion of the considered system can systematically be identified by deriving the corresponding Lagrangian $\mathcal{L}$. As $\vec{B} = \nabla \times \vec{A}$ with e.g. $\vec{A} = xB\vec{e}_y$ as a gauge choice, the Lagrangian is given by:

$$\mathcal{L}(\vec{r}, \vec{v}) = \frac{1}{2}mv^2 + Fx + q\vec{v} \cdot \vec{A} = \frac{1}{2}mv^2 + Fx + qv_y xB.$$

As $\mathcal{L}$ is independent of the coordinate y, the conjugate momentum p_y is an invariant:

$$p_y = \frac{\partial \mathcal{L}}{\partial v_y} = mv_y + qxB = \text{const} \Longrightarrow X = x + \frac{v_y}{\Omega} = \text{const},$$

where $\Omega = qB/m$ is the gyro-frequency, and X is simply the position along Ox of the gyro-centre (or guiding centre) of the particle (note that $-v_y/\Omega$ is the projection of the Larmor radius $\lambda_L \sim (\vec{v} \times \vec{e}_z)/\Omega$ along Ox).

As the fields $\vec{B}$ and $\vec{F}$ are time invariant, the Hamiltonian H, i.e. the energy, of the system is also a constant of motion:

$$H = \vec{v} \cdot \frac{\partial \mathcal{L}}{\partial \vec{v}} - \mathcal{L} = \vec{v} \cdot (m\vec{v} + q\vec{A}) - \frac{1}{2}mv^2 - Fx - q\vec{v} \cdot \vec{A} = \frac{1}{2}mv^2 - Fx.$$

The equilibrium distribution for each species can thus in general be written:

$$f_0 = f_0(X, H).$$

The fact that f_0 can be function of X enables a stationary state with inhomogeneities in the Ox direction. The dependence in x of H through the potential term $-Fx$ naturally leads to inhomogeneities along Ox as well.

Assuming the system is near thermodynamic equilibrium, one considers the quasi-Maxwellian:

$$f_0(X, H) = \frac{\mathcal{N}(X)}{[2\pi T(X)/m]^{3/2}} \exp\left(-\frac{H}{T(X)}\right), \tag{1.5}$$

where $T(x)$ defines the local temperature, and $\mathcal{N}(x)$, as shown below, is related to the local density.

One assumes in the following that the inhomogeneities related to $\mathcal{N}(x)$, $T(x)$, and $\vec{F}$ are weak compared to the Larmor radii $\lambda_L = v_{th}/\Omega$ (the thermal velocity is defined here by $v_{th}^2 = T/m$). In other words, one assumes that the characteristic length L of variations of the equilibrium profiles are such that $\epsilon = \lambda_L/L \ll 1$. This enables to expand f_0 to lowest order in this small parameter:

$$f_0(X, H) = f_0(x, H) + \frac{\partial f_0(x, H)}{\partial x}\frac{v_y}{\Omega} + \mathcal{O}(\epsilon^2), \tag{1.6}$$

$$\frac{\partial f_0}{\partial x} = \left[\frac{d \ln \mathcal{N}}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right] f_0 = \left[\frac{d \ln \mathcal{N}}{dx} + \frac{d \ln T}{dx}\left(\frac{H}{T} - \frac{3}{2}\right)\right] f_0. \tag{1.7}$$

Note, that on the right hand side of Eq.(1.6) f_0 can now be considered as a function of x instead of $X = x + v_y/\Omega$.

Using Eq.(1.6) to compute the **density** N:

$$N(x) = \int d\vec{v}\, f_0(X, H) \simeq \int d\vec{v}\, f_0(x, H) + \underbrace{\int d\vec{v}\, \frac{\partial f_0}{\partial x}\frac{v_y}{\Omega}}_{=0} = \mathcal{N}(x) \exp\left(\frac{Fx}{T(x)}\right),$$

which clearly shows the relation between the density N and the function $\mathcal{N}$ (one naturally has $N \equiv \mathcal{N}$ if $\vec{F} = 0$). From this last relation one obtains:

$$\left.\frac{d\ln N}{dx}\right|_{x=0} = \left.\frac{d\ln \mathcal{N}}{dx}\right|_{x=0} + \frac{F}{T(x=0)}. \tag{1.8}$$

The condition $\lambda_L/L \ll 1$ of weak inhomogeneities thus in particular implies:

$$\left|\lambda_L \frac{F}{T}\right| = \left|\frac{F}{qB}\frac{1}{v_{th}}\right| = \frac{|\vec{v}_F|}{v_{th}} \ll 1, \tag{1.9}$$

where $\vec{v}_F = (\vec{F} \times \vec{B})/(qB^2)$ is the particle drift related to $\vec{F}$.

Using again Eq.(1.6) to compute the **average velocity** $\vec{V}_d$:

$$\begin{aligned}\vec{V}_d(x) &= \frac{1}{N}\int d\vec{v}\,\vec{v} f_0(X, H) \simeq \frac{1}{N}\underbrace{\int d\vec{v}\,\vec{v} f_0(x, H)}_{=0} + \frac{1}{N}\int d\vec{v}\,\vec{v}\,\frac{\partial f_0(x, H)}{\partial x}\frac{v_y}{\Omega} \\ &= \frac{1}{N}\left[\frac{d\ln\mathcal{N}}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right]\int d\vec{v}\,\vec{v} f_0(x, H)\frac{v_y}{\Omega} = \frac{1}{N}\left[\frac{d\ln\mathcal{N}}{dx} + \frac{dT}{dx}\frac{\partial}{\partial T}\right]\underbrace{\mathcal{N}(x)\exp\left(\frac{Fx}{T(x)}\right)}_{N(x)}\frac{v_{th}^2}{\Omega}\vec{e}_y,\end{aligned}$$

so that at $x = 0$:

$$\begin{aligned}\vec{V}_d(x=0) &= \frac{T}{qB}\left[\left.\frac{d\ln\mathcal{N}}{dx}\right|_{x=0} + \left.\frac{d\ln T}{dx}\right|_{x=0}\right]\vec{e}_y = \frac{T}{qB}\left[\left.\frac{d\ln N}{dx}\right|_{x=0} + \left.\frac{d\ln T}{dx}\right|_{x=0} - \frac{F}{T}\right]\vec{e}_y \\ &= \frac{1}{qB^2}\left(-\frac{\nabla p}{N} + \vec{F}\right) \times \vec{B},\end{aligned}$$

having used Eq.(1.8). The average velocity is thus the superposition of the diamagnetic drift due to the pressure gradient ∇p, with $p = NT$, and the particle drift $\vec{v}_F = v_F\,\vec{e}_y$ related to the force $\vec{F}$, with $v_F = -F/qB$.

1.2.2 Solving the Linearised Vlasov Equation

One now assumes that the system is perturbed by an electrostatic fluctuation ϕ. As the unperturbed system is homogeneous in the Oy and Oz directions, as well as in time, and assuming small perturbations, one may consider linear perturbations of the form:

$$\phi = \hat{\phi}(x) \exp i(k_y y + k_z z - \omega t).$$

Due to the constraints on the particle motion in the Ox direction, enabling only excursions of the order of the Larmor radius λ_L, the coupling along Ox is weak. One can thus consider perturbations local to the surface $x = 0$, and in the following one will omit the x dependence of $\hat{\phi}$. Also, for the following derivation, the quantities $\mathcal{N}$, N and T, as well as their gradients are understood to be evaluated at $x = 0$.

The perturbation δf of each particle distribution $f = f_0 + \delta f$ has a similar dependence:

$$\delta f = \hat{\delta f} \exp i(k_y y + k_z z - \omega t).$$

The fluctuation δf is solution of the linearised Vlasov equation:

$$\left.\frac{D}{Dt}\right|_{u.t.p.} \delta f = \left[\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} + \frac{1}{m}\left(q\vec{v}\times\vec{B} + \vec{F}\right)\cdot\frac{\partial}{\partial \vec{v}}\right]\delta f = \frac{q}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial \vec{v}}. \tag{1.10}$$

where $\left.\frac{D}{Dt}\right|_{u.t.p.}$ stands for the total time derivative along the unperturbed trajectories. The equation (1.10) can thus be solved for δf by **integrating along these unperturbed trajectories of the particles**:

$$\delta f(\vec{r}, \vec{v}, t) = \frac{q}{m}\int_{-\infty}^{t} dt' \ \nabla\phi\cdot\left.\frac{\partial f_0}{\partial \vec{v}}\right|_{\vec{r}'(t'),\vec{v}'(t'),t'}, \tag{1.11}$$

having assumed $\mathrm{Im}(\omega) > 0$ to impose causality, so that $\delta f(t = -\infty) = 0$. At the end of this derivation, one may analytically prolong the relations into the half-plane $\mathrm{Im}(\omega) < 0$ to consider possible damped modes.

In Eq.(1.11), the unperturbed particle trajectories $[\vec{r}'(t'), \vec{v}'(t')]$ are thus defined by:

$$\begin{aligned} \frac{d\vec{r}'}{dt'} &= \vec{v}', \\ \frac{d\vec{v}'}{dt'} &= \frac{1}{m}\left(q\vec{v}'\times\vec{B} + \vec{F}\right), \end{aligned}$$

with the initial conditions:

$$\vec{r}'(t' = t) = \vec{r}, \qquad \vec{v}'(t' = t) = \vec{v}.$$

These trajectories can easily be integrated as follows:

$$\vec{r}'(t') = \vec{r} + \frac{1}{\Omega}\mathbf{Q}(t' - t)\,(\vec{v} - \vec{v}_F) + \vec{v}_F\,(t' - t), \tag{1.12}$$

$$\vec{v}'(t') = \mathbf{R}(t' - t)\,(\vec{v} - \vec{v}_F) + \vec{v}_F, \tag{1.13}$$

with the matrices $\mathbf{Q}$ and $\mathbf{R}$ defined by:

$$\begin{aligned} \mathbf{Q}(\tau) &= \begin{pmatrix} \sin(\Omega\tau) & -[\cos(\Omega\tau) - 1] & 0 \\ \cos(\Omega\tau) - 1 & \sin(\Omega\tau) & 0 \\ 0 & 0 & \Omega\tau \end{pmatrix}, \\ \mathbf{R}(\tau) &= \frac{1}{\Omega}\frac{d}{d\tau}\mathbf{Q} = \begin{pmatrix} \cos(\Omega\tau) & \sin(\Omega\tau) & 0 \\ -\sin(\Omega\tau) & \cos(\Omega\tau) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

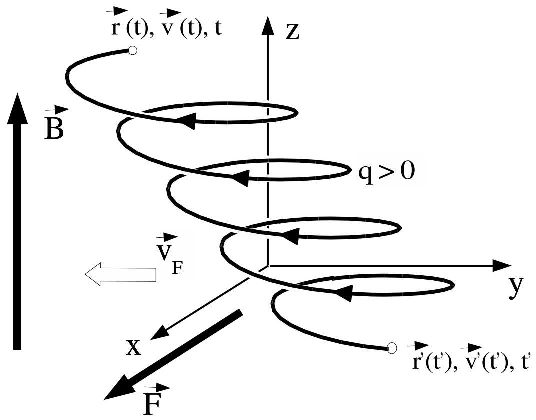

Figure 1.5: To solve the linearised Vlasov equation, and obtain the value of the perturbation $\delta f(\vec{r}, \vec{v}, t)$ at the point $(\vec{r}, \vec{v})$ in phase space at time t, one integrates along the trajectory $[\vec{r}\,'(t'), \vec{v}\,'(t'), t']$ of the magnetised particle submitted to an external force $\vec{F}$. The trajectory is such that $\vec{r}\,'(t' = t) = \vec{r}$ and $\vec{v}\,'(t' = t) = \vec{v}$.

Equations (1.12)-(1.13) clearly show that the particle trajectory is the superposition of a gyro-motion around $\vec{B}$ and a drift $\vec{v}_F$ perpendicular to $\vec{B}$ (see Fig. 1.5). Starting to explicit the integrand of (1.11), one has

$$\frac{\partial f_0}{\partial \vec{v}} = \left[\frac{\vec{e}_y}{\Omega} \left(\frac{d \ln \mathcal{N}}{dx} + \frac{dT}{dx} \frac{\partial}{\partial T} \right) - \frac{\vec{v}}{v_{th}^2} \right] f_0,$$

having again expanded to lowest order in Larmor. Noting that $\nabla \phi = i \vec{k} \phi$, with $\vec{k} = k_y \vec{e}_y + \vec{k}_z \vec{e}_z$, one obtains

$$\nabla \phi \cdot \frac{\partial f_0}{\partial \vec{v}} = i \left[\frac{k_y}{\Omega} \left(\frac{d \ln \mathcal{N}}{dx} + \frac{dT}{dx} \frac{\partial}{\partial T} \right) - \frac{\vec{k} \cdot \vec{v}}{v_{th}^2} \right] f_0 \phi = \frac{1}{v_{th}^2} \left[i \omega_d' - i \vec{k} \cdot \vec{v} \right] f_0 \, \hat{\phi} \, \exp i(\vec{k} \cdot \vec{r} - \omega t),$$

having defined the **drift frequency** operator (contains partial derivative with respect to T):

$$\begin{aligned} \omega_d' &= \frac{T k_y}{qB} \left(\frac{d \ln \mathcal{N}}{dx} + \frac{dT}{dx} \frac{\partial}{\partial T} \right) = \frac{T k_y}{qB} \left(\frac{d \ln N}{dx} + \frac{dT}{dx} \frac{\partial}{\partial T} - \frac{F}{T} \right) \\ &= \omega_N + \omega_T' + \omega_F \\ &\sim \vec{k} \cdot \vec{V}_d, \end{aligned}$$

with

$$\begin{aligned} \omega_N &= \frac{T k_y}{qB} \frac{d \ln N}{dx}, \\ \omega_T' &= \frac{T k_y}{qB} \frac{dT}{dx} \frac{\partial}{\partial T}, \\ \omega_F &= \vec{k} \cdot \vec{v}_F = -\frac{k_y F}{qB}. \end{aligned}$$

Here the prime superscript has been used for pointing out that ω_d' and ω_T' are operators.

Further noting that $df_0(\vec{r}', \vec{v}', t')/dt' = 0$, as f_0 is a stationary state, so that

$$\frac{d}{dt'}\left[f_0\hat{\phi}\,\exp i(\vec{k}\cdot\vec{r}' - \omega t')\right] = i(\vec{k}\cdot\vec{v}' - \omega)f_0\,\hat{\phi}\,\exp i(\vec{k}\cdot\vec{r}' - \omega t'),$$

the integrand of Eq.(1.11) can now be written:

$$\nabla\phi\cdot\left.\frac{\partial f_0}{\partial\vec{v}}\right|_{\vec{r}'(t'),\vec{v}'(t'),t'} = \frac{1}{v_{th}^2}\left[i(\omega_d' - \omega) - \frac{d}{dt'}\right] f_0\,\hat{\phi}\,\exp i(\vec{k}\cdot\vec{r}' - \omega t').$$

Using this last equation, and integrating by parts, the relation (1.11) for δf now becomes

$$\hat{\delta f} = \delta f\exp -i(\vec{k}\cdot\vec{r} - \omega t) = -\frac{q\hat{\phi}}{T}\left\{1 - i(\omega_d' - \omega)\int_{-\infty}^{t} dt'\,\exp i\left[\vec{k}\cdot(\vec{r}' - \vec{r}) - \omega(t' - t)\right]\right\} f_0,$$

the integration by parts having revealed the **adiabatic contribution** $-q\hat{\phi}f_0/T$.

The time integration of the phase factor is carried out as follows:

$$\begin{aligned}
\int_{-\infty}^{t} dt'\,e^{i\left[\vec{k}\cdot(\vec{r}'-\vec{r})-\omega(t'-t)\right]} &= \int_{-\infty}^{0} d\tau\,e^{i\left\{\frac{k_y}{\Omega}[v_x(\cos\Omega\tau - 1) + (v_y - v_F)\sin\Omega\tau + v_F\Omega\tau] + k_z v_z\tau - \omega\tau\right\}} \\
&= \int_{-\infty}^{0} d\tau\,e^{i\frac{k_y v_\perp}{\Omega}\sin(\Omega\tau+\theta)}e^{-i\frac{k_y v_\perp}{\Omega}\sin\theta}e^{i(k_z v_z + \omega_F - \omega)\tau} \\
&= \sum_{n,n'=-\infty}^{+\infty} J_n(\frac{k_y v_\perp}{\Omega})J_{n'}(\frac{k_y v_\perp}{\Omega})\,e^{i(n-n')\theta}\int_{-\infty}^{0} d\tau\,e^{i(k_z v_z + n\Omega + \omega_F - \omega)\tau} \\
&= \sum_{n,n'=-\infty}^{+\infty} \frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right)J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right)e^{i(n-n')\theta}}{i(k_z v_z + n\Omega + \omega_F - \omega)},
\end{aligned}$$

having used Eq.(1.12) for $\vec{r}'$, $\tau = t - t'$, as well as the Fourier decomposition of $\exp(i\,z\sin\theta)$ in terms of Bessel functions of the first kind $J_n(z)$:

$$e^{i\,z\sin\theta} = \sum_{n=-\infty}^{+\infty} J_n(z)e^{in\theta},$$

and having defined the variables $v_\perp$ and θ such that

$$v_x = v_\perp\sin\theta, \qquad v_y - v_F = v_\perp\cos\theta. \tag{1.14}$$

The amplitude of the distribution fluctuation can thus finally be written:

$$\hat{\delta f} = -\frac{q\hat{\phi}}{T}\left[1 - (\omega_d' - \omega)\sum_{n,n'=-\infty}^{+\infty}\frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right)J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right)e^{i(n-n')\theta}}{k_z v_z + n\Omega + \omega_F - \omega}\right] f_0, \tag{1.15}$$

1.2.3 Computing the Dielectric Function

The **dispersion relation** is obtained from the Poisson equation written in Fourier representation:

$$-\triangle\phi = k^2\phi = \frac{1}{\epsilon_0}\sum_{\text{species}} q\delta N$$

$$\Longrightarrow \quad \epsilon(\vec{k},\omega) \doteq 1 - \sum_{\text{species}} \frac{q}{\epsilon_0 k^2}\frac{\delta\hat{N}}{\hat{\phi}} = 0, \tag{1.16}$$

where $\delta\hat{N}$ is the amplitude of the density fluctuation for a given species, and $\epsilon(\vec{k},\omega)$ is the **dielectric function**.

To obtain the dielectric function at the surface $x = 0$, one must therefore integrate $\hat{\delta f}$, given by Eq.(1.15), over velocities to obtain the density fluctuation amplitude:

$$\begin{aligned}\delta\hat{N} &= \int d\vec{v}\,\hat{\delta f}\\ &= -N\frac{q\hat{\phi}}{T}\left[1-(\omega_d'-\omega)\sum_{n,n'=-\infty}^{+\infty}\int d\vec{v}\,\frac{f_0}{N}\frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right)J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right)e^{i(n-n')\theta}}{k_z v_z + n\Omega + \omega_F - \omega}\right].\end{aligned} \tag{1.17}$$

In this last relation, one can consider

$$f_0 = f_0(x=0,\vec{v}) = \frac{N}{(2\pi v_{th}^2)^{3/2}}\exp -\frac{1}{2}\frac{v^2}{v_{th}^2} \simeq \frac{N}{(2\pi v_{th}^2)^{3/2}}\exp -\frac{1}{2}(\frac{v_\perp^2}{v_{th}^2}+\frac{v_z^2}{v_{th}^2}),$$

having used Eq.(1.5), Eq.(1.14) as well as (1.9). The integral $\int d\vec{v}$ in Eq.(1.17) can thus be separated into the integrals $\int d\theta$, $\int v_\perp dv_\perp$, and $\int dv_z$ as follows:

$$\begin{aligned}\delta\hat{N} &= -N\frac{q\hat{\phi}}{T}\left[1-(\omega_d'-\omega)\sum_{n=-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\int\frac{dv_z}{v_{th}}\frac{e^{-\frac{1}{2}\frac{v_z^2}{v_{th}^2}}}{k_z v_z-(\omega-\omega_F-n\Omega)}\int\frac{v_\perp dv_\perp}{v_{th}^2}J_n^2\left(\frac{k_y v_\perp}{\Omega}\right)e^{-\frac{1}{2}\frac{v_\perp^2}{v_{th}^2}}\right]\\ &= -N\frac{q\hat{\phi}}{T}\left\{1-(\omega_d'-\omega)\sum_{n=-\infty}^{+\infty}\frac{1}{\omega-\omega_F-n\Omega}\left[W\left(\frac{\omega-\omega_F-n\Omega}{|k_z|v_{th}}\right)-1\right]\Lambda_n(\xi)\right\},\end{aligned} \tag{1.18}$$

with $\xi = (k_y v_{th}/\Omega)^2 = (k_y\lambda_L)^2$. The integral over v_z has thus been expressed in terms of the dispersion function $W(z)$:

$$W(z) = \frac{1}{\sqrt{2\pi}}\int_\Gamma dx\frac{x}{x-z}\exp(-x^2/2), \tag{1.19}$$

which in particular accounts here for the wave-particle resonances along the magnetic field lines. In Eq.(1.19) the integral path Γ is taken from $x=-\infty$ to $x=\infty$ and, consistent with causality, avoids the pole $x = z$ from below. The integral over $v_\perp$ has been expressed in terms of the scaled, modified Bessel function $\Lambda_n(x) = \exp(-x)I_n(x)$, having used the relation:[5]

$$\int_0^{+\infty} x\,dx\,\exp(-\rho^2x^2)J_p(\alpha x)J_p(\beta x) = \frac{1}{2\rho^2}\exp(-\frac{\alpha^2+\beta^2}{2})I_p(\frac{\alpha\beta}{2\rho^2}).$$

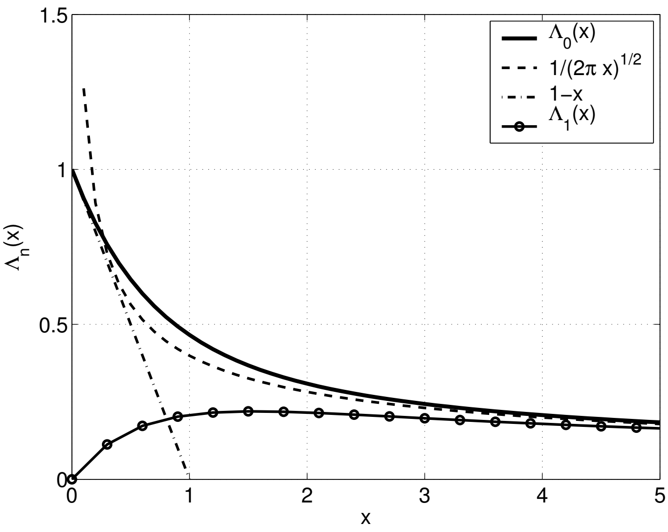

Figure 1.6: Scaled modified Bessel functions $\Lambda_n(\xi) = \exp(-\xi)I_n(\xi)$, for $n = 0, 1$, as well as linear approximation $\Lambda_0(\xi) \simeq 1 - \xi$ for $\xi \ll 1$ and asymptotic relation $\Lambda_0(\xi) \simeq 1/\sqrt{2\pi\xi}$ for $\xi \to \infty$.

These scaled, modified Bessel functions represent the finite Larmor radius effects of the particle. These functions are plotted for $n = 0$ and $n = 1$ in Fig.1.6.

Finally, inserting Eq.(1.18) into Eq.(1.16), one obtains for the dielectric function:

$$\epsilon(\vec{k}, \omega) = 1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + (\omega - \omega_d') \sum_{n=-\infty}^{+\infty} \frac{1}{\omega - \omega_F - n\Omega} \left[W\left(\frac{\omega - \omega_F - n\Omega}{|k_z| v_{th}} \right) - 1 \right] \Lambda_n(\xi) \right\}, \tag{1.20}$$

where the Debye length is defined by $\lambda_D^2 = v_{th}^2/\omega_p^2 = \epsilon_0 T/(N q^2)$.

In the following, one essentially considers the low frequency, long wavelength modes, as they are the most relevant with respect to turbulent transport. One thus assumes the limit $|\omega| \ll |\Omega_{e,i}|$, so that only the zero order ($n = 0$) cyclotron harmonic needs to be retained in Eq.(1.20). In this case, the dispersion relation $\epsilon(\vec{k}, \omega) = 0$ reduces to:

$$\epsilon(\vec{k}, \omega) = 1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + \frac{\omega - \omega_d'}{\omega - \omega_F} \left[W\left(\frac{\omega - \omega_F}{|k_z| v_{th}} \right) - 1 \right] \Lambda_0(\xi) \right\} = 0. \tag{1.21}$$

Exercise:

1. Consider various limits of the dielectric function $\epsilon(\vec{k}, \omega)$ given by Eq.(1.20). In particular, consider the limit of zero gradients and $\vec{F} = 0$, as well as the limit of zero magnetic field $\vec{B}$.

2. Derive and solve the dispersion relation for Electron Plasma Waves (EPWs), as well as sound (= Ion Acoustic Waves, IAW) in the case of an *homogeneous* magnetised plasma.

1.3 The Flute Instability

The flute instability (also called Rayleigh-Taylor or Interchange instability) arises from a charge sign independent force $\vec{F}$, such as a gravitational field, acting against the density gradient of a plasma supported by a magnetic field. This instability is similar to the instability of a heavy liquid supported by a light liquid against a gravitational field. The basic mechanism of the flute instability is illustrated in Fig. 1.7.

To analyse this scenario, one solves the dispersion relation (1.21), in the particular case where the perturbation is transverse to the magnetic field, i.e. $\vec{k} = k_y \vec{e}_y$.

The fact that $k_z = 0$ implies that the particles cannot interact resonantly with the perturbation mode through their motion along the magnetic field lines. Mathematically, this is reflected by the dispersion function term $W(z)$ going to zero for all species in Eq(1.21). Physically, the instability that arises from such a situation can thus be considered of hydrodynamic type.

Furthermore, one shall assume here that the system presents only a density gradient, but no temperature gradient. The drift frequency thus only contains the terms $\omega_d' = \omega_N + \omega_F$.

Finally, one considers wavelengths such that one can have $\xi_i = (k_y \lambda_{L\,i})^2 \simeq 1$, but as a result of the low mass ratio $m_e/m_i \ll 1$ one has $\xi_e = (k_y \lambda_{L\,e})^2 \ll 1$, so that $\Lambda_0(\xi_e) \simeq 1$.

The dispersion relation with the contributions from electrons and ions can thus be written:

$$1 + \frac{1}{(k\lambda_{De})^2}\left[1 - \frac{\omega - \omega_{de}}{\omega - \omega_{Fe}}\right] + \frac{1}{(k\lambda_{Di})^2}\left[1 - \frac{\omega - \omega_{di}}{\omega - \omega_{Fi}}\Lambda_0(\xi_i)\right] = 0. \tag{1.22}$$

This defines a second order polynomial equation for ω, which (after some algebra) can be expressed as:

$$(\omega - \omega_{Fe})^2 - \left[\omega_{Ni}\frac{1 - \Lambda_0}{(k\lambda_{Di})^2 + 1 - \Lambda_0} - \omega_{Fe} + \omega_{Fi}\right](\omega - \omega_{Fe}) + \omega_{Ni}\frac{\omega_{Fi} - \omega_{Fe}}{(k\lambda_{Di})^2 + 1 - \Lambda_0} = 0, \tag{1.23}$$

having made use of the relation $\omega_{Ni} = -(T_i/T_e)\omega_{Ne}$ (reminding that one always assumes $Z = 1$) and the notation $\Lambda_0 = \Lambda_0(\xi_i)$. Using the definitions

$$\mu = 1 - \frac{\omega_{Fe}}{\omega_{Fi}},$$
$$\nu = \frac{\omega_{Ni}}{\omega_{Fi}}\frac{1}{(k\lambda_{Di})^2 + 1 - \Lambda_0},$$

equation (1.23) can be cast into the more compact form:

$$\left(\frac{\omega - \omega_{Fe}}{\omega_{Fi}}\right)^2 - [\mu + \nu(1 - \Lambda_0)]\frac{\omega - \omega_{Fe}}{\omega_{Fi}} + \mu\nu = 0,$$

whose two solutions are given by

$$\frac{\omega - \omega_{Fe}}{\omega_{Fi}} = \frac{1}{2}\left\{\mu + \nu(1 - \Lambda_0) \pm \left([\mu + \nu(1 - \Lambda_0)]^2 - 4\mu\nu\right)^{1/2}\right\}. \tag{1.24}$$

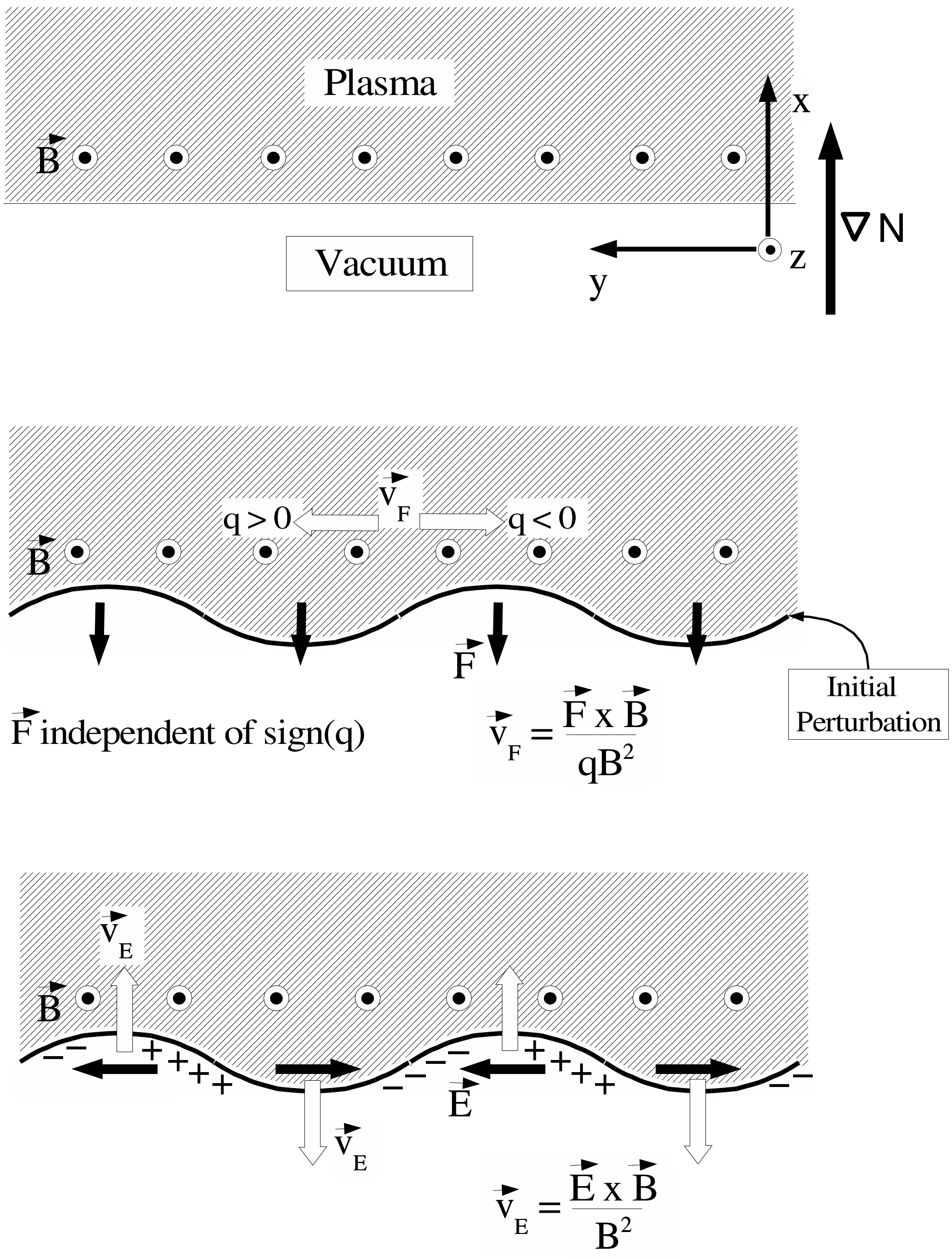

Figure 1.7: Basic mechanism of the flute instability: One considers a magnetised plasma with density gradient ∇N, and submitted to a charge independent force $\vec{F}$ perpendicular to $\vec{B}$. In the presence of an initial density perturbation perpendicular to $\vec{B}$, the drifts $\vec{v}_F = (\vec{F} \times \vec{B})/(qB^2)$ lead to a charge separation. The corresponding electric field $\vec{E}$ induces charge independent drifts $\vec{v}_E = (\vec{E} \times \vec{B})/B^2$, which amplify the perturbation, thus leading to an instability.

The **condition for instability** thus reads:

$$4\mu\nu > [\mu + \nu(1-\Lambda_0)]^2 , \qquad (1.25)$$

in which case one has a growth rate:

$$\gamma = \frac{|\omega_{Fi}|}{2}\left(4\mu\nu - [\mu + \nu(1-\Lambda_0)]^2\right)^{1/2} .$$

Note, that as the right hand side of Eq.(1.25) is positive, a necessary (but insufficient) condition for an instability to arise is to have $\mu\nu > 0$. Furthermore, assuming that the forces $\vec{F}_e$ and $\vec{F}_i$ are oriented in the same direction, i.e. the orientation of the forces $\vec{F}$ is independent of the sign of the particle charge (as in the case of a gravitational force e.g.), the drift frequencies ω_{Fe} and ω_{Fi} have opposite sign, and thus $\mu > 0$. The necessary (but not sufficient) condition thus becomes:

$$\nu > 0 \quad \Longrightarrow \quad \frac{\omega_{Ni}}{\omega_{Fi}} > 0 \quad \Longrightarrow \quad F_i \frac{d\ln N}{dx} < 0, \qquad (1.26)$$

having used the fact that the denominator $[(k\lambda_{Di})^2 + 1 - \Lambda_0]$ in ν is positive definite. This last relation clearly points out that the forces $\vec{F}$ must oppose the density gradient for the instability to arise.

1.3.1 Case of Gravitational Field

Here one considers $F_e = -m_e g$, and $F_i = -m_i g$. The orientation for the gravitational field $\vec{g}$ has been chosen such that the forces $\vec{F} = m\vec{g}$ oppose the density gradient, assumed such that $d\ln N/dx > 0$.

In this case one obtains:

$$\begin{aligned}
\mu &= 1 + \frac{m_e}{m_i} \simeq 1, \\
\nu &= \frac{T_i}{m_i g}\frac{d\ln N}{dx}\frac{1}{(k\lambda_{Di})^2 + 1 - \Lambda_0} \simeq \frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}\frac{1}{\xi_i} \\
\nu(1-\Lambda_0) &\simeq \frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx},
\end{aligned}$$

having assumed sufficiently long wavelengths such that $\xi_i = (k\lambda_{L\,i})^2 \ll 1$, as well as physical parameters such that $\lambda_{L\,i} \gg \lambda_{Di}$. The instability condition (1.25) can then be written:

$$4\frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}\frac{1}{\xi_i} > \left[1 + \frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}\right]^2 \quad \Longrightarrow \quad (k\lambda_{L\,i})^2 < 4\frac{\frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}}{\left[1 + \frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}\right]^2}.$$

This instability condition requires $g(d\ln N/dx) > 0$, which is naturally a particular case of Eq.(1.26). This last relation also clearly illustrates that the instability can only arise at sufficiently long wavelengths compared to the ion Larmor radii. This reflects the so-called Larmor radius stabilisation effect.

In the limit of long wavelength ($\Rightarrow 4\mu\nu \gg [\mu + \nu(1 - \Lambda_0)]^2$), one obtains from Eq. (1.24) the following frequency and growth rate:

$$\omega = \frac{gk}{2\Omega_i}\left(1 + \frac{v_{th\,i}^2}{g}\frac{d\ln N}{dx}\right) + i\left(g\frac{d\ln N}{dx}\right)^{1/2}.$$

1.3.2 Case of Gradient and Curvature of Magnetic Field

Although the dielectric function Eq.(1.20) has been derived assuming a uniform magnetic field $\vec{B}$, it can nonetheless be applied for studying, at least in a qualitative way, the behaviour of a magnetised plasma for which the field $\vec{B}$ presents gradients and curvature.

For the purpose of the present illustration, let us therefore give here a brief description of the forces acting on the particles in such a situation. In the presence of gradients ∇B of the magnetic field, the magnetic moment $\mu = mv_\perp^2/(2B)$, related to the gyro-motion of the charged particle, is submitted to a force $\vec{F}_\mu = -\mu\nabla_\perp B$. In the presence of curvature of the magnetic field, the particle is submitted to the centrifugal force $\vec{F}_c = -mv_\parallel^2\vec{e}_\parallel \cdot (\nabla\vec{e}_\parallel)$. Here $v_\perp$ and $v_\parallel$ are respectively the components of the velocity perpendicular and parallel to $\vec{B}$, and $\vec{e}_\parallel = \vec{B}/B$. Note that the forces $\vec{F}_\mu$ and $\vec{F}_c$ are both charge sign independent and can thus give rise to a flute instability. One can show, that in the case of a low pressure plasma, one has $\vec{e}_\parallel \cdot (\nabla\vec{e}_\parallel) = \nabla_\perp \ln B$, and these two forces can thus be combined as follows:

$$\vec{F} = \vec{F}_\mu + \vec{F}_c = -m\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\nabla_\perp \ln B.$$

This force is dependent on the velocity of the particle. However, for the purpose of our simple slab model of the flute instability, one considers an average of this force, the average being taken over the particle distribution, so that one takes:

$$\vec{F} \leftarrow <\vec{F}> = \int d\vec{v} f_0 \vec{F} / \int dv f_0 = -2T\nabla_\perp \ln B. \tag{1.27}$$

Defining $R = |\nabla_\perp \ln B|^{-1}$ the gradient length of the magnetic field, one has

$$F = -\frac{2T}{R},$$

where the sign has again been chosen such that $\vec{F}$ opposes the density gradient, which is the necessary condition for the flute instability. Considering Eq.(1.27), this requires the magnetic and density gradient to have the same orientation. In terms of the curvature, this corresponds to a convex geometry of the magnetic field confining the plasma. Note that this average force is now independent of the particle mass, by opposition to the gravitational force.

For the flute instability analysis, one obtains in this case:

$$\begin{aligned}\mu &= 1+\frac{T_e}{T_i},\\ \nu &= \frac{R}{2}\frac{d\ln N}{dx}\frac{1}{(k\lambda_{Di})^2+1-\Lambda_0}\simeq\frac{R}{2}\frac{d\ln N}{dx}\frac{1}{\xi_i}\\ \nu(1-\Lambda_0) &\simeq \frac{R}{2}\frac{d\ln N}{dx},\end{aligned}$$

having again assumed $\xi_i = (k\lambda_{L\,i})^2 \ll 1$ (long wavelengths), as well as $\lambda_{L\,i} \gg \lambda_{Di}$. The instability condition (1.25) then becomes:

$$4\left(1+\frac{T_e}{T_i}\right)\frac{R}{2}\frac{d\ln N}{dx}\frac{1}{\xi_i} > \left[1+\frac{T_e}{T_i}+\frac{R}{2}\frac{d\ln N}{dx}\right]^2 \quad\Longrightarrow\quad (k\lambda_{L\,i})^2 < 4\frac{\frac{R}{2}\frac{d\ln N}{dx}\left(1+\frac{T_e}{T_i}\right)}{\left[1+\frac{T_e}{T_i}+\frac{R}{2}\frac{d\ln N}{dx}\right]^2}.$$

As for the gravitational case, note the lower limit on the wavelengths, as well as the effect of the Larmor radius stabilisation.

Finally, in the long wavelength limit, the growth rate becomes in this case:

$$\gamma = \omega_{Fi}(\mu\nu)^{1/2} = \left[\frac{T_e+T_i}{m_i}\frac{2}{R}\frac{d\ln N}{dx}\right]^{1/2}.$$

1.4 The Drift Wave Instability

The presence of an "external" force $\vec{F}$, such as for the flute instability, is not required for an instability to arise in an inhomogeneous plasma. One shows here, that the presence of a density gradient alone is sufficient for the onset of an instability.

Let us first show how a density gradient enables a wave to propagate essentially perpendicular to the magnetic field $\vec{B}$, at a phase velocity of the order of the diamagnetic drift velocity of the electrons $\vec{V}_{Ne} = (T_e/eB^2)\nabla \ln N \times \vec{B}$. This is the so-called drift wave. The basic mechanism of propagation of the drift wave is illustrated in Fig. 1.8.

Despite the fact that the propagation is considered mainly perpendicular to the magnetic field $\vec{B}$ in this case (i.e. $|k_z/k_y| \ll 1$), one assumes that the phase velocity along the magnetic field is nonetheless sufficiently low so that the electrons can respond adiabatically, which is the case if $|\omega/(k_z v_{th\,e})| \ll 1$. For the ions however one still assumes $|\omega/(k_z v_{th\,i})| \gg 1$. These are similar conditions for the existence of the sound wave (= Ion Acoustic Wave, IAW) in an homogeneous magnetised plasma. Actually, as shown below, the drift wave appears as the low frequency "deformation" of the sound wave.

1.4.1 Two-fluid model of the Drift Wave

A first model for the drift wave can thus be given by the following two-fluid description:

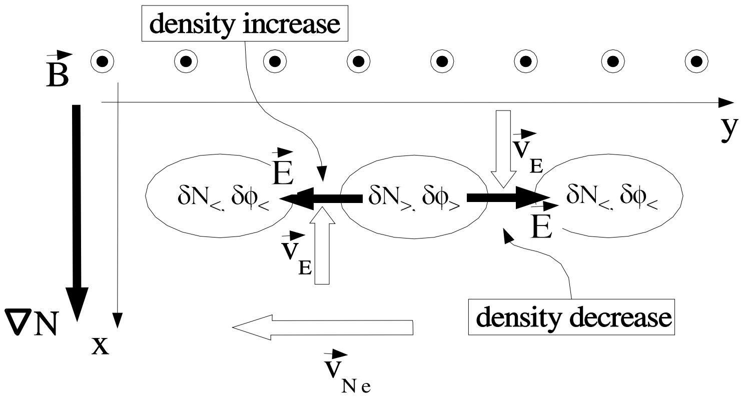

Figure 1.8: Basic Mechanism of the drift wave: One considers a magnetised plasma with just a density gradient ∇N in its equilibrium state, and submitted to a perturbation which is quasi-perpendicular to $\vec{B}$. Assuming that the electrons may respond adiabatically and ensure quasineutrality, the density fluctuation $\delta N = \delta N_i = \delta N_e$ and associated potential field ϕ are in phase. From the viewpoint considered in this illustration ($\vec{B}$ pointing upward) the resulting convection $\vec{v}_E = (\vec{E}\times\vec{B})/B^2$ of the plasma is clockwise (resp. counter-clockwise) oriented around a maximum $\delta N_>$ (resp. minimum $\delta N_<$) in the density fluctuation. Given the orientation of the density gradient in the figure, this leads to a density increase (resp. decrease) to the left of a maximum (resp. minimum) in the density fluctuation. The perturbation thus propagates to the left here, i.e. in the same direction as the electron diamagnetic current $\vec{V}_{Ne} = (T/eB^2)\nabla \ln N\times\vec{B}$.

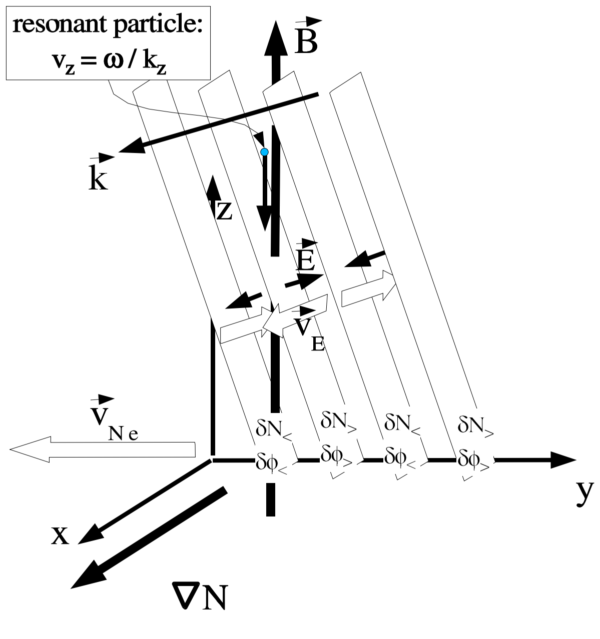

Figure 1.9: Same situation as in figure 1.8 but showing how the magnetised particles, travelling essentially along the magnetic field $\vec{B}$, can interact resonantly if their parallel velocity v_z is near the parallel phase velocity ω/k_z of the wave. These resonant particles can not only lead to Landau damping of the wave as in an homogeneous plasma, but can be destabilising as well. This destabilising effect results from the fact that these particles undergo an essentially constant $\vec{v}_E$ drift, contrary to the bulk particles undergoing an oscillatory $\vec{v}_E$ drift. This net flow can lead to a reinforcement of the density perturbation and thus to instability.

- Cold ions, represented by the continuity and momentum equation:

$$\begin{aligned}\frac{\partial N_i}{\partial t} + \nabla\cdot(N_i\vec{u}_i) &= 0,\\ m_i\left[\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i\cdot(\nabla\vec{u}_i)\right] &= e\left(\vec{E} + \vec{u}_i\times\vec{B}\right).\end{aligned}$$

- Adiabatic electrons:

$$N_e = N_{e0}\exp(\frac{e\phi}{T_e}).$$

- Assuming low frequency waves ($|\omega/\omega_p| \ll 1$) and sufficiently long wavelengths ($k\lambda_{De} \ll 1$), quasi-neutrality can be considered for closure:

$$N_e = N_i.$$

Linearising these equations for low amplitude electrostatic fluctuations $\vec{E} = -\nabla\phi \sim \exp(-i\omega t)$ in the case of a plasma with gradients ∇N_0 of the equilibrium density $N_0 \equiv N_{e0} = N_{i0}$:

$$\begin{aligned}-i\omega\,\delta N_i = \frac{\partial\,\delta N_i}{\partial t} &= -N_0\nabla\cdot\vec{u}_i - \vec{u}_i\cdot\nabla N_0, &(1.28)\\ -i\omega\, m_i\vec{u}_i = m_i\frac{\partial\vec{u}_i}{\partial t} &= e\left(-\nabla\phi + \vec{u}_i\times\vec{B}\right), &(1.29)\\ \delta N_e &= N_0\frac{e\phi}{T_e}, &(1.30)\\ \delta N_e &= \delta N_i. &(1.31)\end{aligned}$$

The ion momentum equation (1.29) can readily be solved for $\vec{u}_i$:

$$\begin{aligned}\vec{u}_i &= \frac{1}{1-(\Omega_i/\omega)^2}\frac{\Omega_i}{i\omega B}\left[\nabla\phi - \frac{\Omega_i}{i\omega}\nabla\phi\times\frac{\vec{B}}{B} - \left(\frac{\Omega_i}{\omega}\right)^2\nabla_\parallel\phi\right]\\ &\simeq -\frac{1}{B^2}\nabla\phi\times\vec{B} + \frac{1}{i\omega}\frac{e}{m_i}\nabla_\parallel\phi + \frac{i\omega}{\Omega_i}\frac{1}{B}\nabla_\perp\phi, \qquad (1.32)\end{aligned}$$

having made use of the fact that the solution to $\vec{u} = \vec{a} + \vec{u}\times\vec{b}$ is given by $\vec{u} = (\vec{a} + \vec{a}\times\vec{b} + \vec{a}\cdot\vec{b}\,\vec{b})/(1+b^2)$, as well as of the low frequency assumption $|\omega/\Omega_i| \ll 1$. The first term of the solution (1.32) to $\vec{u}_i$ is the $\vec{v}_E = (\vec{E}\times\vec{B})/B^2$ drift, the second term corresponds to the oscillatory motion parallel to $\vec{B}$. The third term is the so-called polarisation drift, which is charge dependant and corresponds to a small oscillatory motion in the same direction as $\nabla_\perp\phi$ (thus, polarisation drift and the $\vec{v}_E$ term are orthogonal).

Inserting Eq. (1.32) into the continuity Eq. (1.28), leads to:

$$\begin{aligned}\frac{\delta N_i}{N_0} &= \frac{1}{i\omega}\left(\nabla\cdot\vec{u}_i + \vec{u}_i\cdot\nabla\ln N_0\right)\\ &= \frac{1}{\Omega_i B}\triangle_\perp\phi - \frac{1}{\omega^2}\frac{e}{m_i}\nabla_\parallel^2\phi + \frac{1}{i\omega}\frac{1}{B^2}(\nabla\ln N_0\times\vec{B})\cdot\nabla\phi, \qquad (1.33)\end{aligned}$$

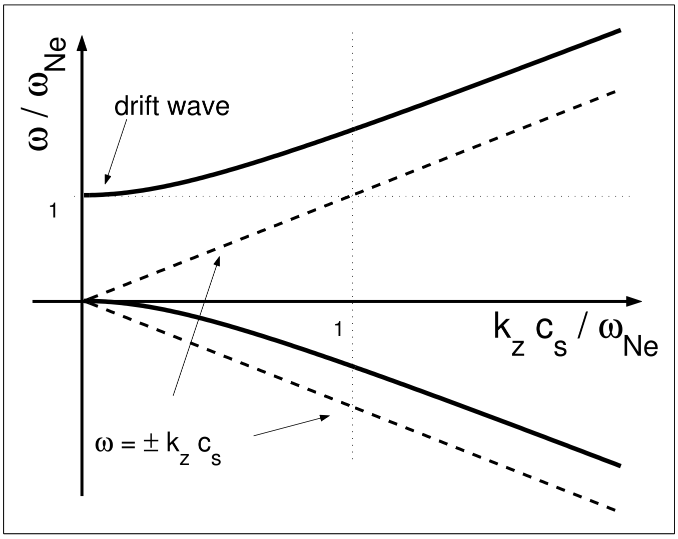

Figure 1.10: Sound branches deformed by the presence of a density gradient ∇N. For low k_z, one of the branches transforms into the drift mode with frequency ω_{Ne}.

having made use of the fact that the wave propagation is such that $\nabla\phi \cdot \nabla N_0 = 0$. Finally, inserting Eqs.(1.30) and (1.33) into (1.31) leads to the equation for ϕ:

$$\left(1 - \rho^{\star 2} \triangle_\perp - \frac{1}{i\omega}\vec{V}_{Ne} \cdot \nabla + \frac{c_s^2}{\omega^2}\nabla_\parallel^2\right)\phi = 0,$$

which for a wave $\phi \sim \exp(i\vec{k} \cdot \vec{r})$, and $\vec{k} = k_y\vec{e}_y + k_z\vec{e}_z$, gives the following dispersion relation:

$$\left[1 + (k_y\rho^\star)^2\right]\omega^2 - \omega_{Ne}\omega - (k_z c_s)^2 = 0.$$

Here, $\omega_{Ne} = \vec{V}_{Ne} \cdot \vec{k}$, and $\rho^\star = c_s/\Omega_i$ is the ion Larmor radius evaluated at the sound speed $c_s = \sqrt{T_e/m_i}$.

In the absence of the density gradient, i.e. $\omega_{Ne} = 0$, this dispersion relation is clearly the one for sound waves in a homogeneous, magnetised plasma (propagation at phase velocity c_s along $\vec{B}$). In the presence of the density gradient, the two sound wave branches are significantly “deformed” for $|k_z c_s| \ll |\omega_{Ne}|$:

$$\omega = \frac{\omega_{Ne} \pm \sqrt{\omega_{Ne}^2 + 4(k_z c_s)^2[1 + (k_y\rho^\star)^2]}}{2[1 + (k_y\rho^\star)^2]} = \frac{\omega_{Ne}}{1 + (k_y\rho^\star)^2} \times \begin{cases} 1 + (k_z c_s/\omega_{Ne})^2[1 + (k_y\rho^\star)^2], \\ -(k_z c_s/\omega_{Ne})^2[1 + (k_y\rho^\star)^2]. \end{cases} \quad (1.34)$$

Thus, for $|k_z c_s| \ll |\omega_{Ne}|$, one of the two sound branches has frequency $\omega \simeq \omega_{Ne}$, and thus propagates quasi-perpendicularly to the magnetic field at the electron diamagnetic drift velocity: $\omega/k_y \simeq V_{Ne}$. This is the so-called drift wave. The two sound branches deformed by the presence of the density gradient are shown in Fig.1.10.

Recall however that these results can not be taken for k_z exactly zero, as otherwise the assumption of adiabatic electron response breaks down. This condition is verified for

$|\omega_{Ne}/k_z| \ll v_{th\,e}$, which imposes a lower limit on the ratio $|k_z/k_y|$, while the condition $|k_z c_s| \ll |\omega_{Ne}|$ provides an upper limit:

$$\sqrt{\frac{m_e}{m_i}} \ll \sqrt{\frac{T_i}{T_e}} \frac{1}{\lambda_{L\,i}|\nabla \ln N|} |k_z/k_y| \ll 1.$$

which determines a clear interval thanks to the small mass ratio m_e/m_i.

The assumption of cold ions, which requires $|\omega_{Ne}/k_z| \gg v_{th\,i}$, leads to the condition:

$$\sqrt{\frac{T_i}{T_e}} \frac{1}{\lambda_{L\,i}|\nabla \ln N|} |k_z/k_y| \ll \sqrt{\frac{T_e}{T_i}},$$

but which is obviously weaker than the one resulting from $|k_z c_s| \ll |\omega_{Ne}|$ if $T_e \geq T_i$.

1.4.2 Kinetic Analysis of the Drift Mode Instability

Let us now analyse how the resonant particles, in fact the electrons, can lead to a destabilisation of the drift wave. For this purpose, the mode is reconsidered in the framework of the kinetic description by solving the dispersion relation Eq.(1.21) in the appropriate limit. More exactly, one considers Eq.(1.21) for $\vec{F}_{e,i} = 0$, $dT_{e,i}/dx = 0$, and $\xi_e = (k_y \lambda_{L\,e})^2 \ll 1$, so that the kinetic dispersion relation becomes:

$$\begin{aligned} 0 = \epsilon(\vec{k}, \omega) = 1 \quad &+ \quad \frac{1}{(k\lambda_{De})^2} \left\{ 1 + (1 - \frac{\omega_{Ne}}{\omega}) \left[W\left(\frac{\omega}{|k_z| v_{th\,e}} \right) - 1 \right] \right\} \\ &+ \quad \frac{1}{(k\lambda_{Di})^2} \left\{ 1 + (1 - \frac{\omega_{Ni}}{\omega}) \left[W\left(\frac{\omega}{|k_z| v_{th\,i}} \right) - 1 \right] \Lambda_0(\xi_i) \right\}. \end{aligned}$$

The dispersion function $W(z)$ can be expanded in the appropriate limits for electrons and ions:

$$|z_e| = |\omega/k_z v_{th\,e}| \ll 1 \quad \Longrightarrow \quad W(z_e) \simeq 1 + i\sqrt{\frac{\pi}{2}} z_e, \tag{1.35}$$

$$|z_i| = |\omega/k_z v_{th\,i}| \gg 1 \quad \Longrightarrow \quad W(z_i) \simeq -\frac{1}{z_i^2} + i\sqrt{\frac{\pi}{2}} z_i \exp(-\frac{1}{2} z_i^2). \tag{1.36}$$

The real frequency $\omega_R \doteq \text{Re}(\omega)$ of the drift mode can be recovered in the resonant approximation by solving:

$$0 = \text{Re}[\epsilon(\vec{k}, \omega_R)] \simeq 1 + \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2} \left\{ 1 - (1 - \frac{\omega_{Ni}}{\omega_R}) \left[1 + (\frac{k_z v_{th\,i}}{\omega_R})^2 \right] \Lambda_0(\xi_i) \right\}.$$

By fully dropping the term in $k_z v_{th\,i}/\omega_R$ (quasi-perpendicular propagation), one indeed obtains:

$$\omega_R = \omega_{Ne} \frac{\Lambda_0(\xi_i)}{1 + (T_e/T_i)[1 - \Lambda_0(\xi_i)] + (k\lambda_{De})^2}. \tag{1.37}$$

Note the finite ion Larmor radius effects – term Λ_0 in numerator, and $(T_e/T_i)(1-\Lambda_0)$ in denominator – as well as the term $(k\lambda_{De})^2$ in the denominator related to the deviation from quasineutrality.

In the limit $\xi_i = (k_y\lambda_{L\,i})^2 \ll 1$ so that $\Lambda_0(\xi_i) \simeq 1-\xi_i$, and $\lambda_{L\,i} \gg \lambda_{D\,e,i}$ Eq.(1.37) leads to:

$$\omega_R \simeq \frac{\omega_{Ne}}{1+(k_y\rho^\star)^2}.$$

One thus has indeed recovered the solution (1.34) from the fluid model. One also notices that the so-called polarisation drift term, which already appeared in the two-fluid model, derives in the kinetic description from the lowest order ion Larmor radius effects represented by $\Lambda_0(\xi_i)$.

In the resonant approximation, the growth rate γ is given by:

$$\gamma = -\left.\frac{\mathrm{Im}(\epsilon)}{\partial\mathrm{Re}(\epsilon)/\partial\omega}\right|_{\omega_R}. \tag{1.38}$$

In this case, one has

$$\mathrm{Im}[\epsilon(\omega_R)] = \frac{1}{(k\lambda_{De})^2}(1-\frac{\omega_{Ne}}{\omega_R})\sqrt{\frac{\pi}{2}}z_e + \frac{\Lambda_0(\xi_i)}{(k\lambda_{Di})^2}(1-\frac{\omega_{Ni}}{\omega_R})\sqrt{\frac{\pi}{2}}z_i\exp(-\frac{z_i^2}{2}), \tag{1.39}$$

$$\frac{\partial\mathrm{Re}[\epsilon(\omega_R)]}{\partial\omega} = -\frac{\Lambda_0(\xi_i)}{(k\lambda_{Di})^2}\frac{\omega_{Ni}}{\omega_R^2}. \tag{1.40}$$

Inserting (1.39) and (1.40) into Eq.(1.38) then leads to:

$$\gamma = \sqrt{\frac{\pi}{2}}\frac{\omega_R^2}{\Lambda_0(\xi_i)}\left\{\left(1-\frac{\omega_R}{\omega_{Ne}}\right)\frac{1}{|k_z|v_{th\,e}} - \left(1+\frac{T_e}{T_i}\frac{\omega_R}{\omega_{Ne}}\right)\frac{\Lambda_0(\xi_i)}{|k_z|v_{th\,i}}\exp-\frac{1}{2}\left(\frac{\omega_R}{k_z v_{th\,i}}\right)^2\right\}.$$

Noticing from Eq.(1.37) that in fact $0 < \omega_R/\omega_{Ne} < 1$, one can conclude that the ion contribution to γ has a damping effect, while the effective electron contribution is destabilising. The stabilising ion contribution being exponentially small, can usually be neglected, so that after insertion of Eq.(1.37) for ω_R the growth rate becomes:

$$\gamma = \sqrt{\frac{\pi}{2}}\frac{\omega_{Ne}^2}{|k_z|v_{th\,e}}\frac{\Lambda_0\left[(1+T_e/T_i)(1-\Lambda_0)+(k\lambda_{De})^2\right]}{\left[1+(T_e/T_i)(1-\Lambda_0)+(k\lambda_{De})^2\right]^3}. \tag{1.41}$$

In the limit $\xi_i = (k_y\lambda_{L\,i})^2 \ll 1$, as well as $\lambda_{L\,i} \gg \lambda_{Di}$, one obtains:

$$\gamma = \sqrt{\frac{\pi}{2}}\frac{\omega_{Ne}^2}{|k_z|v_{th\,e}}\frac{(1+T_e/T_i)(k_y\lambda_{L\,i})^2}{\left[1+(T_e/T_i)(k_y\lambda_{L\,i})^2\right]^3}.$$

From the following derivation, the drift instability is clearly the result of a resonant interaction between the wave and the particles. Note how in Eq.(1.39) the contribution to $\mathrm{Im}[\epsilon(\omega_R)]$ –representing the resonant effects– from each species is multiplied by a factor $(1-\omega_N/\omega_R)$. The term 1 in this factor is related to the Landau damping effect, as already present in an homogeneous plasma. The term $-\omega_N/\omega_R$ in this factor is obviously specific

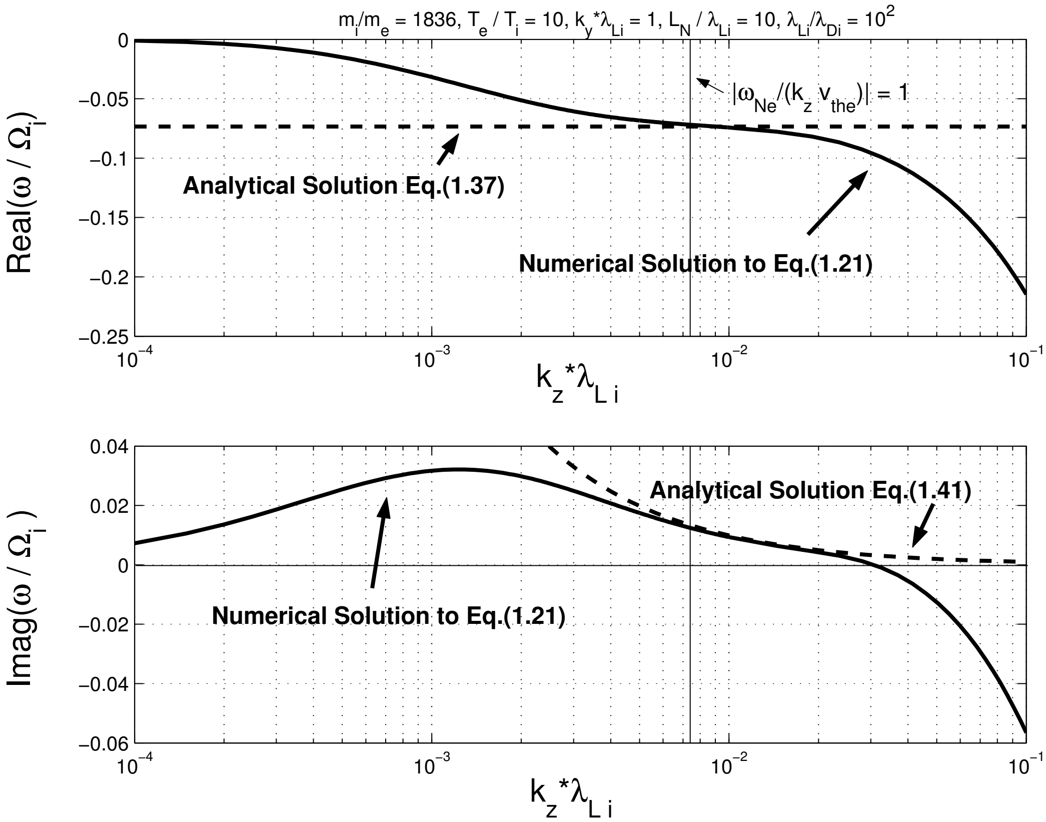

Figure 1.11: Numerical and analytical solutions over a scan in k_z for the real frequency and growth rate of the drift wave instability. Considered parameters are $m_i/m_e = 1836$ (protons), $T_e/T_i = 10$, $k_y\lambda_{L\,i} = 1$, $L_N/\lambda_{L\,i} = 10$, $\lambda_{L\,i}/\lambda_{Di} = 10^2$.

to the inhomogeneous plasma. This term can be negative and thus have a destabilising effect. It represents the resonant convection in the $\vec{v}_E = (\vec{E} \times \vec{B})/B^2$ drift, along or against the density gradient ∇N, of resonant particles, i.e. whose velocity v_z along the magnetic field $\vec{B}$ matches the phase velocity ω/k_z of the mode in this same direction (see Fig. 1.9).

In somewhat more detail, the bulk particles, which do not keep in phase with the mode, slosh back and forth in the passing wave, both in the direction parallel to $\vec{B}$, due to the drive by the parallel component $E_\parallel$ of the perturbation field $\vec{E}$, as well as in the perpendicular direction, due to the drift $\vec{v}_E$. Resonant particles however, which by definition keep in phase with the mode (at least in the linear stage of evolution), undergo a drive by the perturbation field $\vec{E}$ whose orientation does not change. The drive along $\vec{B}$ by $E_\parallel$ of these resonant particles leads to Landau damping and to the related flattening, at $v_z = \omega_R/k_z$, of the velocity distribution $f(v_z)$. The convection $\vec{v}_E$ perpendicular to $\vec{B}$ of the resonant particles results in a reinforcement of the density perturbation and, if this mechanism is sufficiently strong compared to Landau damping, leads in this way to the destabilisation of the wave.

1.5 The Slab Ion Temperature Gradient (Slab-ITG) Instability

It was shown in the previous section how the presence of a density gradient alone can lead to an instability. One shows here how an instability of a different nature can also arise from the presence of a temperature gradient.

1.5.1 Basic Study of the Slab-ITG

One assumes again that one is in the regime $|\omega/(k_z v_{th\,e})| \ll 1$ such that the electrons respond adiabatically, and $k\lambda_{De} \ll 1$ so that quasi-neutrality can be invoked.

One starts by considering a plasma with just an ion temperature gradient $\nabla T_i \neq 0$ (no density gradient, $\nabla N = 0$). From Eq.(1.21), the dispersion relation can then be written:

$$\epsilon(\vec{k}, \omega) = \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2} \left\{ 1 + \left(1 - \frac{\omega'_{T_i}}{\omega}\right) \left[W\left(\frac{\omega}{|k_z| v_{th\,i}}\right) - 1 \right] \Lambda_0(\xi_i) \right\} = 0. \quad (1.42)$$

Furthermore, assuming that the mode is such that $|\omega/(k_z v_{th\,i})| \gg 1$, enabling again to expand the dispersion function according to Eq.(1.36), and neglecting at first the finite ion Larmor radius effects, one obtains:

$$\frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2} \left\{ 1 - \left(1 - \frac{\omega'_{T_i}}{\omega}\right) \left[1 + \left(\frac{k_z v_{th\,i}}{\omega}\right)^2 \right] \right\} = 0.$$

Note that the resonant term $i\sqrt{\pi/2}\, z_i \exp(-z_i^2/2)$ has already been neglected here, as the instability is dominantly of hydrodynamic type. This will appear clearly in the following.

Carrying out the partial derivative with respect to the ion temperature of the operator $\omega'_{T_i} = (T_i k_y/eB)(dT_i/dx)\partial/\partial T_i$ then leads to:

$$1 - \left(\frac{k_z c_s}{\omega}\right)^2 \left(1 - \frac{\omega_{T_i}}{\omega}\right) = 0, \tag{1.43}$$

where now ω_{T_i} without the prime superscript stands for $\omega_{T_i} = (T_i k_y/eB)(d\ln T_i/dx)$.

Note first, that in the absence of the temperature gradient, one obtains $\omega = \pm k_z c_s$, so that the considered mode is again the deformation of one of the sound wave branches. For $\nabla T_i \neq 0$, Eq(1.43) provides a cubic equation for ω, which can have two complex conjugate solutions, i.e. represent an instability. Indeed, assuming $|\omega| \ll |\omega_{T_i}|$ Eq.(1.43) becomes:

$$1 + \frac{\omega_{T_i}(k_z c_s)^2}{\omega^3} = 0,$$

which has a solution with positive imaginary part, i.e. an unstable mode:

$$\omega = (\frac{1}{2} + i\frac{\sqrt{3}}{2})|\omega_{T_i}(k_z c_s)^2|^{1/3}. \tag{1.44}$$

The above assumption $|\omega| \ll |\omega_{T_i}|$ is thus verified if

$$|k_z c_s| \ll |\omega_{T_i}|, \tag{1.45}$$

which in turn imposes an upper limit on the ratio $|k_z/k_y|$:

$$\frac{1}{\lambda_{L\,i}|\nabla \ln T_i|}|k_z/k_y| \ll \sqrt{\frac{T_i}{T_e}}. \tag{1.46}$$

The initial assumptions $v_{th\,i} \ll |\omega/k_z| \ll v_{th\,e}$ naturally impose further constraints on $|k_z/k_y|$:

$$\sqrt{\frac{T_i}{T_e}}\left(\frac{m_e}{m_i}\right)^{3/2} \ll \frac{1}{\lambda_{L\,i}|\nabla \ln T_i|}|k_z/k_y| \ll \frac{T_e}{T_i},$$

having made use of $|\omega| \simeq |\omega_{T_i}(k_z c_s)^2|^{1/3}$ according to Eq.(1.44). For $T_i \simeq T_e$ the condition arising from $|\omega/k_z| \gg v_{th\,i}$ is obviously equivalent to Eq.(1.46).

At the limit of applicability of the result (1.44) with respect to the wavelengths along $\vec{B}$, i.e. taking $k_z c_s \simeq \omega_{T_i}$, one obtains:

$$\gamma \simeq \omega_R \simeq k_z c_s \simeq \omega_{T_i},$$

so that $|\omega/k_z| \simeq v_{th\,i}$ for $T_i \simeq T_e$. Furthermore, in the limit of applicability of the result with respect to the finite ion Larmor radius effects, i.e. taking $\xi_i \simeq 1$, one obtains:

$$\begin{aligned} \gamma &\simeq v_{th\,i}\,|\nabla \ln T_i|\,, \\ k_z &\simeq |\nabla \ln T_i|\,, \qquad \text{for} \qquad T_e \simeq T_i. \end{aligned}$$

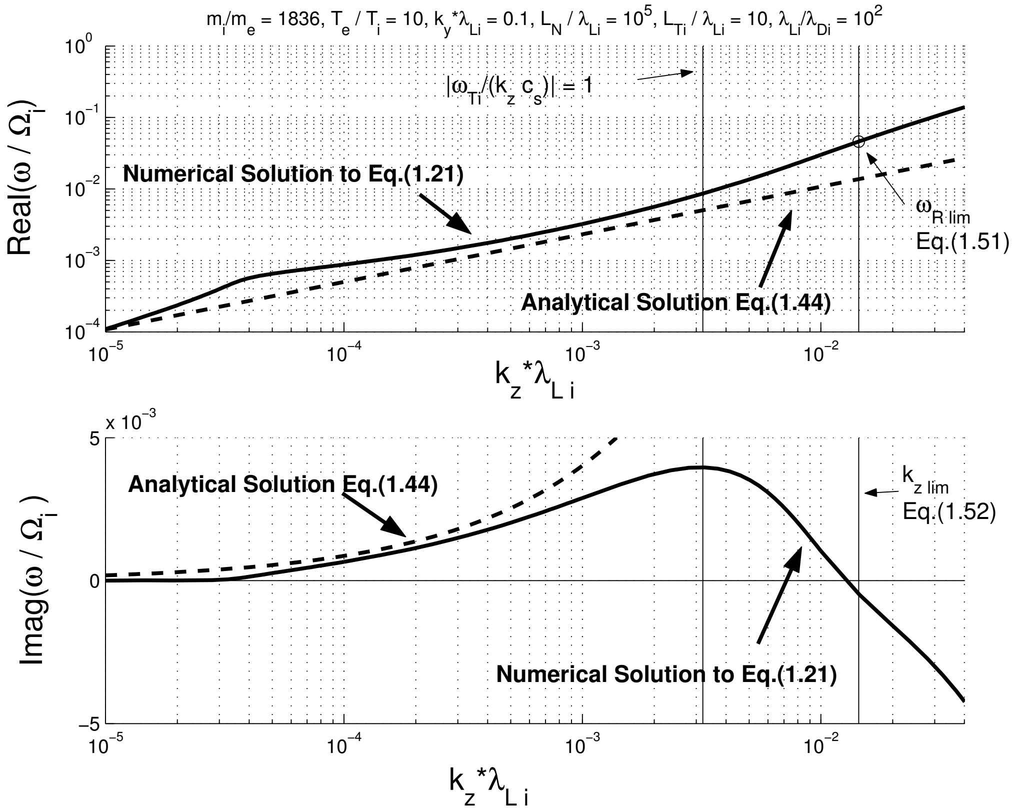

Figure 1.12: Numerical and analytical solutions over a scan in k_z for the real frequency and growth rate of the slab-ITG instability. Considered parameters are $m_i/m_e = 1836$ (protons), $T_e/T_i = 10$, $k_y\lambda_{L\,i} = 0.1$, $L_{T_i}/\lambda_{L\,i} = 10$, $L_N/\lambda_{L\,i} = 10^5$ $\lambda_{L\,i}/\lambda_{Di} = 10^2$.

1.5.2 Instability Boundary for the Slab-ITG

As already shown by the estimate (1.45), the wave vector component k_z must be below a critical value $k_{z\,\text{lim}}$ for the slab-ITG instability to develop. Here this limit is estimated more accurately for arbitrary finite ion Larmor radius effects (ξ_i values).

Furthermore, the slab-ITG instability can also arise under certain conditions where the plasma does not only have an ion temperature gradient, but also a density gradient. The relative importance of the characteristic length L_{T_i} of the ion temperature gradient with respect to the characteristic length L_N of density is measured by the ratio

$$\eta_i \doteq \frac{d\ln T_i}{d\ln N} = \frac{L_N}{L_{T_i}}.$$

The critical values $\eta_{i\,\text{lim}}$ for the slab-ITG instability to develop is derived in the following as well. The dispersion relation (1.42) is thus generalised here to also account for density gradients:

$$\epsilon(\vec{k},\omega) = \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2}\left\{1 + \left(1 - \frac{\omega_{Ni}+\omega'_{T_i}}{\omega}\right)\left[W\left(\frac{\omega}{|k_z|v_{th\,i}}\right) - 1\right]\Lambda_0(\xi_i)\right\} = 0. \tag{1.47}$$

The limits of instability are obtained by finding the conditions under which the solutions ω to the dispersion relation (1.47) are exactly real valued: $\omega = \omega_R \doteq \text{Re}(\omega)$. Indeed, this defines the limit between damped and unstable modes.

One starts by developing Eq.(1.47) by carrying out the derivative with respect to the ion temperature of the operator ω'_{T_i}, making no approximation this time on the dispersion function $W(z)$. Note that ω'_{T_i} operates both on $W(z_i)$ through the $v_{th\,i}$ dependence of its argument $z_i = \omega/|k_z|v_{th\,i}$, as well as on $\Lambda_0(\xi_i)$, as $\xi_i = (k_y v_{th\,i}/\Omega_i)^2$. In this way one obtains for the dispersion relation:

$$\begin{aligned} 0 \;=\; \epsilon(\vec{k},\omega) = &\frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2}\times \qquad (1.48)\\ &\left\{1 + \left(1 - \frac{\omega_{Ni} - \frac{\omega_{T_i}}{2}}{\omega}\right)(W-1)\,\Lambda_0 - \frac{\omega_{T_i}}{\omega}\left[\frac{z_i^2}{2}W\Lambda_0 + (W-1)\,\xi_i\,(\Lambda_1 - \Lambda_0)\right]\right\}, \end{aligned}$$

having used the shorter notations $W = W(z_i)$ and $\Lambda_n = \Lambda_n(\xi_i)$, as well as the relations:

$$\begin{aligned} \frac{dW}{dz} &= \left(\frac{1}{z} - z\right)W - \frac{1}{z},\\ \frac{d\Lambda_0}{d\xi} &= \Lambda_1(\xi) - \Lambda_0(\xi). \end{aligned}$$

To identify the instability boundary, one must thus solve $\text{Re}[\epsilon(\omega_R)] = 0$ and $\text{Im}[\epsilon(\omega_R)] = 0$ using Eq.(1.48). Using the following expression for $W(z)$ in terms of the complex error function:

$$W(z) = 1 + i\sqrt{\frac{\pi}{2}}\,z\,e^{-z^2/2}\left(1 + i\sqrt{\frac{2}{\pi}}\int_0^z dz' e^{z'^2/2}\right),$$

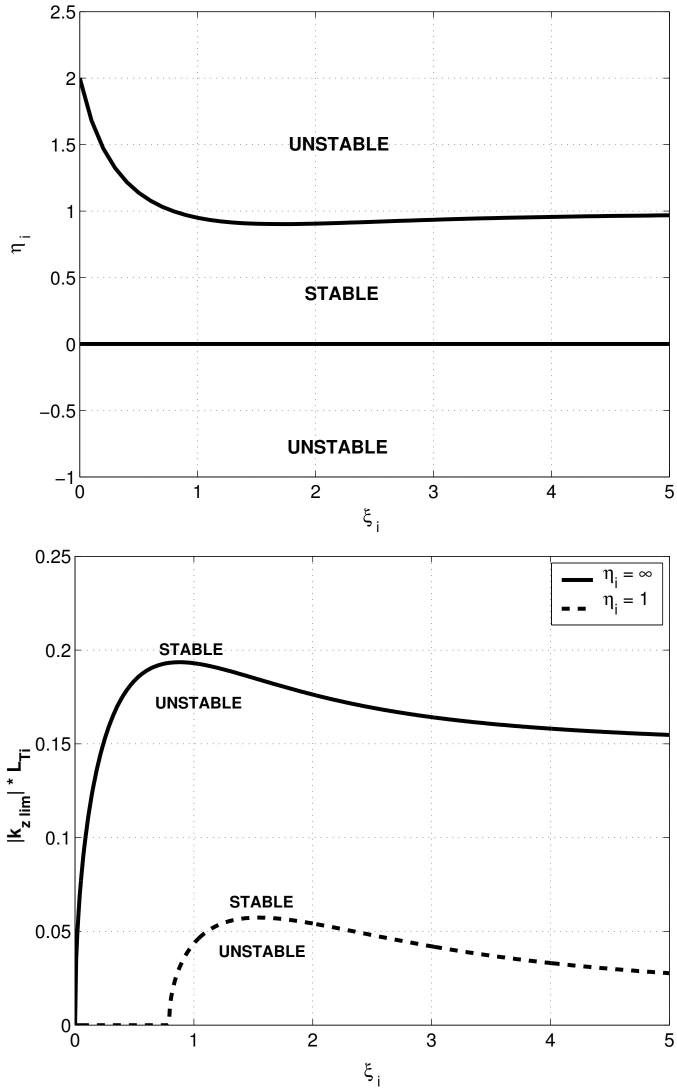

Figure 1.13: Slab-ITG instability: a) Critical η_i value as a function of ξ_i. b) Upper limit on $|k_z|$ as a function of ξ_i for $\eta_i = \infty$ and $\eta_i = 1$. For this figure one assumed $T_e = T_i$.

one obtains, after some minor algebra, the following set of equations for ω_R and k_z as a function of the parameters η_i and ξ_i:

$$\frac{T_i}{T_e} + 1 - \left[1 - \frac{\omega_{Ni}(1-\eta_i/2)}{\omega_R}\right]\Lambda_0 + \frac{\omega_{T_i}}{\omega_R}\,\xi_i\,(\Lambda_1 - \Lambda_0) \;=\; 0, \tag{1.49}$$

$$z_i^2 = \frac{2(1+T_i/T_e)}{\Lambda_0}\frac{\omega_R}{\omega_{T_i}}, \tag{1.50}$$

noting that $\omega_{T_i}/\omega_{Ni} = \eta_i$. Equation (1.49) provides ω_R at the instability boundary:

$$\frac{\omega_R}{\omega_{T_i}} = \frac{1}{2}\frac{\Lambda_0}{T_i/T_e + 1 - \Lambda_0}\left[1 - \frac{2}{\eta_i} - 2\xi_i\frac{\Lambda_1 - \Lambda_0}{\Lambda_0}\right]. \tag{1.51}$$

According to Eq.(1.50) one must have $\omega_R/\omega_{T_i} > 0$. From Eq.(1.51), this in turn imposes a condition for instability on η_i:

$$\begin{aligned} \text{Either} \quad \eta_i &< 0, \\ \text{or} \quad \eta_i &> 2\left(1 - 2\xi_i\frac{\Lambda_1 - \Lambda_0}{\Lambda_0}\right)^{-1}. \end{aligned}$$

These critical values of η_i as a function of ξ_i are plotted in Fig.1.13.

Inserting Eq.(1.51) in Eq.(1.50) for $z_i^2 = \omega_R^2/(k_z v_{th\,i})^2$ finally provides the limiting condition on k_z:

$$|k_z v_{th\,i}| < |k_{z\,\lim}\, v_{th\,i}| = \frac{\omega_{T_i}}{2}\frac{\Lambda_0}{(1+T_i/T_e)^{1/2}(T_i/T_e + 1 - \Lambda_0)^{1/2}}\left[1 - \frac{2}{\eta_i} - 2\xi_i\frac{\Lambda_1 - \Lambda_0}{\Lambda_0}\right]^{1/2}. \tag{1.52}$$

1.5.3 Two-Fluid Model of the slab-ITG Instability

The underlying mechanism of the slab-ITG instability results from heat convection in the presence of an ion temperature gradient. This is illustrated by the following two-fluid model:

- "Hot" ions, represented by the continuity equation, the momentum equation (including the pressure term), and a heat equation representing convection in the flow $\vec{v}_E = (\vec{E} \times \vec{B})/B^2$:

$$\begin{aligned} \frac{\partial N_i}{\partial t} + \nabla \cdot (N_i \vec{u}_i) &= 0, \\ m_i N_i \left[\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot (\nabla \vec{u}_i)\right] &= e N_i \left(\vec{E} + \vec{u}_i \times \vec{B}\right) - \nabla(N_i T_i). \\ \frac{\partial}{\partial t}(N_i T_i) + \nabla \cdot (N_i T_i \vec{v}_E) &= 0. \end{aligned}$$

- Adiabatic electrons:
$$N_e = N_{e0}\exp(\frac{e\phi}{T_e}).$$

- Quasi-neutrality for closure:
$$N_e = N_i.$$

One considers a plasma such that $\nabla N_0 = 0$, and $\nabla T_i \neq 0$. Linearising for small amplitude perturbations $\vec{E} = -\nabla\phi \sim \exp(i\vec{k}\cdot\vec{r} - \omega t)$, and assuming $|\omega/\Omega_i| \ll 1$, this set of equations can be reduced to the following:

$$\begin{aligned}
\frac{\partial\,\delta N_i}{\partial t} + N_0\nabla_\parallel u_{i\,\parallel} &= 0,\\
m_i\frac{\partial u_{i\,\parallel}}{\partial t} &= -e\nabla_\parallel\phi - \nabla_\parallel\delta T_i,\\
\frac{\partial\delta T_i}{\partial t} + \vec{v}_E\cdot\nabla T_i &= 0,\\
\delta N_i = \delta N_e &= N_0\frac{e\phi}{T_e},
\end{aligned}$$

with $\vec{v}_E = (-\nabla\phi\times\vec{B})/B^2$.

Exercise: Show that this last set of equations leads to the same dispersion relation as given by Eq.(1.43).

1.6 Electron Temperature Gradient (ETG) Instability

At sufficiently short perpendicular wavelengths k_y, such that $\xi_i = (k_y\lambda_{L\,i})^2 \gg 1$, the perpendicular perturbation field felt by the ions is averaged out over their gyro-motion. As a result, the response of these particles becomes adiabatic-like. Indeed, for $\xi \to \infty$ one has $\Lambda_n(\xi) \to \exp[-n^2/(2\xi)]/\sqrt{2\pi\xi} \to 0$, and thus from Eq.(1.20) one sees that the ion contribution to the dielectric function reduces to $1/(k\lambda_{Di})^2$, i.e. the adiabatic term. One considers here such a short wavelength regime, however still assuming being in the limit such that $k\lambda_{De} \ll 1$ so that quasineutrality can be invoked. The simultaneous conditions $\xi_i \gg 1$ and $k\lambda_{De} \ll 1$ is possible if $\lambda_{De} \ll \lambda_{L\,i}$, which is the case at least for magnetic fusion-type plasmas (see table 2).

Considering the dispersion relation (1.21) in this limit, and allowing for finite electron Larmor radius effects ($\xi_e \sim 1$), one obtains:

$$\epsilon(\vec{k},\omega) = \frac{1}{(k\lambda_{Di})^2} + \frac{1}{(k\lambda_{De})^2}\left\{1 + \left(1 - \frac{\omega_{Ne} + \omega'_{T_e}}{\omega}\right)\left[W\left(\frac{\omega}{|k_z|v_{th\,e}}\right) - 1\right]\Lambda_0(\xi_e)\right\} = 0, \tag{1.53}$$

which is exactly the same relation as Eq.(1.47) for the slab-ITG, but simply with the electron and ion subscripts interchanged. The results obtained in Sec. 1.5.1 for the slab-ITG can thus be directly translated here to the ETG instability.

In particular, considering at first the limit $\xi_e \ll 1$, and $\eta_e \to \infty$ ($\nabla N = 0$), as well as assuming $|\omega/(k_z v_{th\,e})| \gg 1$ one obtains the simple dispersion relation:

$$1 - \left(\frac{k_z}{\omega}\right)^2 \frac{T_i}{m_e}\left(1 - \frac{\omega_{T_e}}{\omega}\right) = 0, \tag{1.54}$$

which is the equivalent to Eq.(1.43). For sufficiently long parallel wavelengths such that $|k_z\sqrt{T_i/m_e}| \ll |\omega_{Te}|$, Eq. (1.54) again provides an unstable mode with growth rate:

$$\gamma = \frac{\sqrt{3}}{2}|\omega_{T_e} k_z^2 T_i/m_e|^{1/3}. \tag{1.55}$$

At the limit of applicability of this result with respect to the limit on k_z, i.e. taking $|k_z\sqrt{T_i/m_e}| \simeq |\omega_{Te}|$, one obtains:

$$\gamma \simeq k_z\sqrt{T_i/m_e} \simeq \omega_{Te},$$

and at the limit of applicability with respect to short perpendicular wavelengths, i.e. taking $\xi_e \simeq 1$:

$$\begin{aligned}\gamma &\simeq v_{th\,e}\,|\nabla \ln T_e|\,,\\ k_z &\simeq |\nabla \ln T_e|\,, \qquad \text{for} \qquad T_e \simeq T_i.\end{aligned}$$

This last limit may only be considered if $\lambda_{L\,e} > \lambda_{De}$ ($\Leftrightarrow \omega_{pe} > \Omega_e$). Otherwise the limit $k\lambda_{De} \simeq 1$ is met before $\xi_e \simeq 1$, in which case quasi-neutrality is not preserved, and the validity of the dispersion relation Eq.(1.54) breaks down. Thus, in the case $\omega_{pe} < \Omega_e$, at the limit of applicability of Eq.(1.55) for $k\lambda_{De} \simeq 1$:

$$\begin{aligned}\gamma &\simeq \frac{\omega_{pe}}{\Omega_e} v_{th\,e}\,|\nabla \ln T_e|\,,\\ k_z &\simeq \frac{\omega_{pe}}{\Omega_e}\,|\nabla \ln T_e|\,, \qquad \text{for} \qquad T_e \simeq T_i.\end{aligned}$$

1.7 The Toroidal Ion Temperature Gradient (Toroidal-ITG) Instability

If in addition to an ion temperature gradient the plasma is submitted to "external" forces $\vec{F}$ related to gradients and/or curvature of the confining magnetic field $\vec{B}$, the slab-ITG instability presented in Sec. 1.5 can acquire an interchange-like character. One distinguishes this new form of the instability as the so-called toroidal-ITG.

1.7.1 Dispersion Relation

To obtain a relevant local dispersion relation for the toroidal-ITG instability, which correctly accounts for the effects of the effective forces $\vec{F}$ related to the curvature and gradient of the magnetic field $\vec{B}$, one must reconsider the actual validity in this case of the dispersion relation defined by Eq.(1.21). Indeed, in deriving Eq.(1.21), one assumed a plasma confined by a *uniform* magnetic field $\vec{B}$, in which the particles of each species was submitted to a *constant* external force $\vec{F}$.

First, let us note again that the forces $\vec{F}_c$ and $\vec{F}_\mu$ related to the curvature and gradient respectively of the magnetic field are velocity dependent, as already discussed in Sec. 1.3.2. Thus, these forces vary from one particle to another from a given species distribution. Correctly taking account of this velocity dependence is actually essential near marginal stability of the mode, where resonant particle effects are important. We shall again consider here a low pressure plasma (i.e. low β = plasma pressure / magnetic pressure), so that the considered effective force $\vec{F}$ is of the form:

$$\vec{F} = \vec{F}_\mu + \vec{F}_c = -m\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\nabla_\perp \ln B. \tag{1.56}$$

Notice as well, that these forces do not in fact contribute to the energy H of the system as considered in Sec. 1.2.1 by the contribution $-Fx$ to H for a truly external force $\vec{F} = F\vec{e}_x$. Indeed, in a system where particles are only submitted to a magnetic field, the energy reduces to the kinetic energy $H = mv^2/2$.

In view of the above comments, we shall attempt to accordingly correct the derivation of the dispersion relation in Sec.1.2 for the purpose of studying the toroidal-ITG instability. We shall "salvage" the derivation at the level of Eq.(1.15), i.e. the relation for $\hat{\delta f}$, solution to the linearised Vlasov equation. What needs to be done is to reconsider the various terms related to the force $\vec{F}$.

The force $\vec{F}$ appears in Eq.(1.15) for $\hat{\delta f}$ through the drift frequency ω_F, both in the total drift frequency term $\omega_d' = \omega_N + \omega_T' + \omega_F$, as well as in the resonant denominator in the form of a Doppler shift. The ω_F contribution to ω_d' can easily be traced back to the $\vec{F}$ dependence of H, and consequently shall be dropped here. However, the Doppler shift in the resonant denominator is directly related to the $\vec{v}_F$ drifts of the particle trajectories. Such trajectory drifts definitely also result from forces of the form (1.56), as is systematically shown in the framework of Guiding Centre theory.[6] This term is thus retained here, and the relation for the amplitude $\hat{\delta f}$ becomes:

$$\hat{\delta f} = -\frac{q\hat{\phi}}{T}\left[1 - (\omega_N + \omega_T' - \omega)\sum_{n,n'=-\infty}^{+\infty}\frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right)J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right)e^{i(n-n')\theta}}{k_z v_z + n\Omega + \omega_F - \omega}\right]f_0, \tag{1.57}$$

with

$$\omega_F = \vec{k}\cdot\vec{v}_F = -\frac{k_y F}{qB} = \frac{k_y}{qB}\frac{m}{R}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right),$$

and $R = |\nabla_\perp \ln B|^{-1}$. One can easily convince oneself, by reconsidering the derivation of Eq.(1.15) in Sec.1.2, that the existence and form of the Doppler shift in the resonant denominator is indeed independent of the fact that the forces are velocity dependent.

To re-derive the dielectric function $\epsilon(\vec{k}, \omega)$ defined as

$$\epsilon(\vec{k}, \omega) = 1 - \sum_{\text{species}}\frac{1}{(k\lambda_D)^2}\frac{\delta\hat{N}}{N}\frac{T}{q\hat{\phi}},$$

requires to recompute the amplitude of density fluctuations $\delta\hat{N}$ from Eq.(1.57) for $\hat{\delta f}$. This leads to the relevant dispersion relation for studying the toroidal-ITG:

$$\epsilon(\vec{k},\omega) = 1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2}\left[1 + (\omega - \omega_N - \omega_T')\int d\vec{v}\,\frac{f_0}{N}\,\frac{J_0^2\left(\frac{k_y v_\perp}{\Omega}\right)}{k_z v_z + \omega_F - \omega}\right] = 0, \qquad (1.58)$$

with

$$\frac{f_0}{N} = \frac{1}{(2\pi v_{th}^2)^{3/2}}\exp -\frac{1}{2}\left(\frac{v}{v_{th}}\right)^2,$$

and $d\vec{v} = v_\perp dv_\perp dv_z$, and having again only retained the lowest order cyclotron harmonic $n = 0$ under the assumption $|\omega/\Omega| \ll 1$.

Furthermore, again assuming an adiabatic response of electrons, and sufficiently long wavelengths such that $k\lambda_D \ll 1$ so that quasi-neutrality can be invoked, the dispersion relation (1.58) actually reduces to

$$\epsilon(\vec{k},\omega) = \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2}\left[1 + (\omega - \omega_{Ni} - \omega_{Ti}')\int d\vec{v}\,\frac{f_{0i}}{N}\,\frac{J_0^2\left(\frac{k_y v_\perp}{\Omega}\right)}{k_z v_z + \omega_{Fi}(v_\perp, v_z) - \omega}\right] = 0. \qquad (1.59)$$

By the notation $\omega_{Fi}(v_\perp, v_z)$, one has highlighted in this last relation the velocity dependence of the Doppler shift ω_F.

1.7.2 Fluid-Like Limit

Without further approximations, the velocity integral in Eq.(1.59) cannot be expressed in terms of well-known special functions as was the case in Sec.1.2 when considering a constant force $\vec{F}$.

To nonetheless get a first analytical understanding of the toroidal-ITG instability, one considers again a fluid-like limit for the ions by assuming $|\omega/(k_z v_{th\,i})| \gg 1$, and $|\omega/\omega_F| \gg 1$, as well as lowest order finite Larmor radius effects by assuming $k_y \lambda_{L\,i} \ll 1$. In this limit, the integrand of the Maxwellian-weighted velocity integral in Eq.(1.59) can be expanded as follows:

$$\begin{aligned}\frac{J_0^2\left(\frac{k_y v_\perp}{\Omega}\right)}{k_z v_z + \omega_{Fi}(v_\perp, v_z) - \omega} &= -\frac{1}{\omega}\left[1 + \frac{k_z v_z}{\omega} + \left(\frac{k_z v_z}{\omega}\right)^2 + \frac{\omega_{Fi}}{\omega} + \ldots\right] \times \left[1 - \frac{1}{4}\left(\frac{k_y v_\perp}{\Omega}\right)^2 + \ldots\right]^2 \\ &\simeq -\frac{1}{\omega}\left[1 + \frac{k_z v_z}{\omega} + \left(\frac{k_z v_z}{\omega}\right)^2 + \frac{\omega_{Fi}}{\omega} - \frac{1}{2}\left(\frac{k_y v_\perp}{\Omega}\right)^2\right], \qquad (1.60)\end{aligned}$$

having kept the lower order terms of the Taylor expansions $(1 + x)^{-1} = 1 - x + x^2 + \ldots$ and $J_0(x) = 1 - x^2/4 + \ldots$. The velocity integration in Eq.(1.59) for this approximate integrand is now straightforward to carry out, noticing that the average $< v_i >$ over the Maxwellian distribution f_0/N of any velocity coordinate v_i is zero, while the average

$< v_i^2 >$ of the square of any velocity component provides v_{th}^2:

$$\frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2}\left\{1 - \left(1 - \frac{\omega_{Ni} + \omega'_{Ti}}{\omega}\right)\left[1 + \left(\frac{k_z v_{th\,i}}{\omega}\right)^2 + \frac{< \omega_{Fi} >}{\omega} - (k_y \lambda_{L\,i})^2\right]\right\} = 0, \tag{1.61}$$

with

$$< \omega_{Fi} >= \frac{2T_i}{eB}\frac{k_y}{R}.$$

Finally, carrying out the temperature derivative of $\omega'_{Ti} = \omega_{Ti} T_i \partial/\partial T_i$ gives:

$$\frac{T_i}{T_e} + \frac{\omega_{Ni}}{\omega} - \left(1 - \frac{\omega_{Ni} + \omega_{Ti}}{\omega}\right)\left[\left(\frac{k_z v_{th\,i}}{\omega}\right)^2 + \frac{< \omega_{Fi} >}{\omega} - (k_y \lambda_{L\,i})^2\right] = 0. \tag{1.62}$$

One now takes various limits of the dispersion relation defined by Eq.(1.62). One starts by considering the case of an homogeneous plasma ($\Rightarrow \omega_N, \omega_{Ti} = 0$) confined by a uniform magnetic field ($\Rightarrow < \omega_{Fi} >= 0$), so that Eq.(1.62) becomes:

$$\omega^2 = \frac{(k_z c_s)^2}{1 + (k_y \rho^\star)^2},$$

which, as expected, are the two sound branches in a magnetised plasma, including the effect of polarisation drift.

Then, considering the case of a finite ion temperature gradient ($\Rightarrow \omega_{Ti} \neq 0$), however still no density gradient ($\Rightarrow \omega_N = 0$), and neglecting all finite Larmor radius effects, leads to:

$$1 - \left(1 - \frac{\omega_{Ti}}{\omega}\right)\left[\left(\frac{k_z c_s}{\omega}\right)^2 + \frac{T_e}{T_i}\frac{< \omega_{Fi} >}{\omega}\right] = 0. \tag{1.63}$$

Equation (1.63) obviously represents the deformation by the curvature and gradient of the magnetic field ($< \omega_{Fi} >\neq 0$) of the fluid-like dispersion relation for the slab-ITG given by Eq.(1.43). In fact, contrary to Eq.(1.43), equation (1.63) provides an instability with finite growth rate even in the limit $k_z \to 0$. Indeed, assuming again $|\omega| \ll |\omega_{Ti}|$, Eq.(1.63) for $k_z = 0$ becomes:

$$1 + \frac{T_e}{T_i}\frac{\omega_{Ti} < \omega_{Fi} >}{\omega^2} = 0,$$

which has solutions:

$$\omega = \pm\left(-\frac{T_e}{T_i}\,\omega_{Ti} < \omega_{Fi} >\right)^{1/2}. \tag{1.64}$$

Noticing that

$$-\omega_{Ti} < \omega_{Fi} >= \frac{k_y^2}{(eB)^2}\nabla T_i \cdot < \vec{F}_i >= -2\frac{(k_y T_i)^2}{(eB)^2}\nabla \ln T_i \cdot \nabla \ln B,$$

having used $< \vec{F}_i >= -2T_i \nabla \ln B$, the necessary condition for solution (1.64) to provide an unstable mode is for the gradient of the magnetic field ∇B to be in the same direction as the ion temperature gradient ∇T_i ($\rightarrow$ convex magnetised plasma geometry). If ∇B is opposite to ∇T_i ($\rightarrow$ concave magnetised plasma geometry), the mode is stable. This clearly illustrates the interchange-like character of the toroidal-ITG mode. In a low β tokamak plasma, the so-called favourable curvature, i.e. stable with respect to the toroidal-ITG, is thus the inner, high magnetic field region of the torus ($\rightarrow$ concave plasma geometry), while the unfavourable curvature region is the outer, low field region of the torus ($\rightarrow$ convex plasma geometry). The toroidal-ITG thus tends to "balloon" in this outer region.

The relative importance of the slab-like or toroidal-like character of the instability is directly related to the relative importance of the terms $(k_z c_s/\omega)^2$ and $(T_e/T_i)(<\omega_{Fi}>/\omega)$ respectively in Eq.(1.63). Obviously, the true toroidal-like mode appears in the limit $k_z \rightarrow 0$ and the instability is thus said to align with the magnetic field lines.

1.7.3 Stability Conditions

Finally, considering Eq.(1.62) for $k_z = 0$ and allowing for possible density gradients ($\omega_{Ni} \neq 0$), but still neglecting finite larmor radius effects (i.e. polarisation drift), leads to:

$$\frac{1}{\tau} + \frac{\omega_{Ni}}{\omega} - \left(1 - \frac{\omega_{Ni} + \omega_{Ti}}{\omega}\right) \frac{<\omega_{Fi}>}{\omega} = 0,$$

which reduces to

$$\frac{1}{\tau}\left(\frac{\omega}{\omega_{Ni}}\right)^2 + (1 - 2\epsilon_N)\frac{\omega}{\omega_{Ni}} + 2\epsilon_N(1 + \eta_i) = 0,$$

whose solutions are given by

$$\frac{\omega}{\omega_{Ni}} = \frac{\tau}{2}\left\{2\epsilon_N - 1 \pm \left[(2\epsilon_N - 1)^2 - \frac{8}{\tau}\epsilon_N(1 + \eta_i)\right]^{1/2}\right\}, \qquad (1.65)$$

having used the notations $\tau = T_e/T_i$ and $\epsilon_N = <\omega_F>/(2\omega_N) = L_N/R$.

Equation (1.65) provides the following condition for instability for the toroidal-ITG:

$$8\epsilon_N(1 + \eta_i) > \tau(2\epsilon_N - 1)^2, \qquad (1.66)$$

which is equivalent to

$$\eta_i > \frac{\tau}{8}\frac{(2\epsilon_N - 1)^2}{\epsilon_N} - 1, \quad \text{for} \quad \epsilon_N > 0, \qquad (1.67)$$

$$\eta_i < \frac{\tau}{8}\frac{(2\epsilon_N - 1)^2}{\epsilon_N} - 1, \quad \text{for} \quad \epsilon_N < 0. \qquad (1.68)$$

The fluid-like results obtained above for the toroidal-ITG are naturally only of qualitative value, especially near marginal stability where drift resonances appearing in Eq.(1.59) become important, and thus in particular for characterising the stability conditions. This is shown in Fig.1.14, where condition (1.67) is compared to the marginal stability obtained by solving numerically the kinetic dispersion relation (1.59) for $\tau = 1$, $k_z = 0$ and $k_y \lambda_{L\,i} = 0.3$.

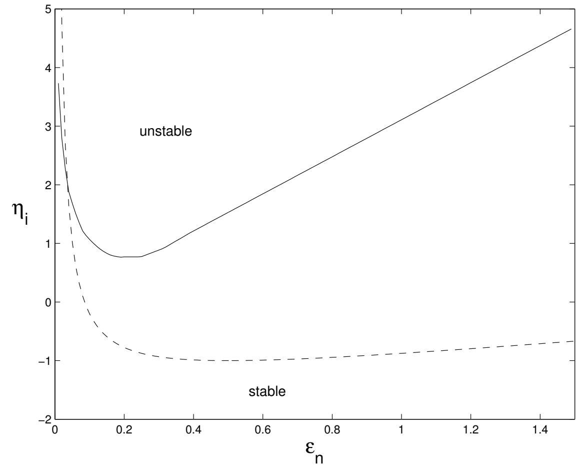

Figure 1.14: Stability curve for toroidal-ITG mode in plane (ϵ_n, η_i). Full line is obtained by numerical resolution of the kinetic dispersion relation Eq.(1.59) for $\tau = 1$, $k_z = 0$, $k_y \lambda_{L\,i} = 0.3$. Dashed line is the fluid result Eq.(1.67).

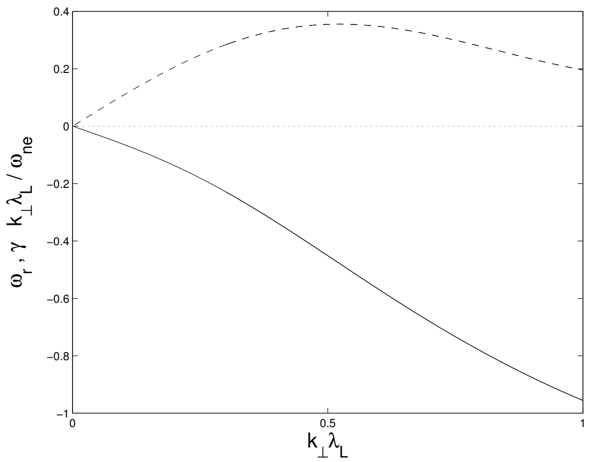

Figure 1.15: Real frequency (full line) and growth rate (dashed line) of toroidal-ITG instability as a function of $k_\perp \lambda_{L\,i}$, obtained by numerically solving Eq.(1.59). Maximum value of growth rate is near $k_\perp \lambda_{L\,i} = 0.5$. Here $\tau = 1$, $\epsilon_N = 0.3$, $\eta_i = 4$ and $k_z = 0$. Note that for $\omega_r/\omega_{Ne} < 0$ the mode propagates in the ion diamagnetic direction.

As can be seen from Eq.(1.66), in the limit of flat density where $\eta_i, \epsilon_N \to \infty$, the condition for instability becomes a constraint on $\epsilon_{T_i} =< \omega_{Fi} > /(2\omega_{Ti}) = L_{Ti}/R$. Numerically solving the full kinetic dispersion relation Eq.(1.59) for $\tau = 1$, one obtains the instability condition $\epsilon_{T_i} \stackrel{\sim}{<} 0.3$ instead of $\epsilon_{T_i} < 2$ coming from the fluid condition Eq.(1.66).

Fig.1.15 illustrates finite Larmor radius effects and shows how the toroidal-ITG has maximum growth rate for $k_y \lambda_{L\,i} \sim 0.5$.

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